Longevity Risk: To Bear or to Insure?

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Abstract

We compare two longevity risk management contracts in retirement: a collective arrangement that distributes the risk among participants, and a market-provided annuity contract. We evaluate the contracts' appeal with respect to the retiree's welfare, and the viability of the market solution through the financial reward to the annuity provider's equity holders. The collective agreement yields marginally higher individual welfare than an annuity contract priced at its best estimate, and the annuity provider is incapable of adequately compensating its equity holders for bearing longevity risk. Therefore, market-provided annuity contracts would not co-exist with collective schemes.

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1 Introduction

Longevity risk is a looming threat to pension systems worldwide. In contrast to mortality risk, which is the idiosyncratic risk surrounding an individual's actual date of death given known survival probabilities, longevity risk is the risk of mis-estimating future survival probabilities.[1] This systematic risk can be distressful for retirement financing because longevity-linked assets are not yet commonplace (Tan et al., 2015).

The global transition of funded pensions from Defined-Benefit (DB) to Defined-Contribution (DC) plans[2] precipitates the need for a sustainable means of managing mortality and longevity risks, which have conventionally been borne by the DB plan sponsor. The essence of a DC setup grants individuals full freedom in managing their retirement capital, which is accumulated at a statutory rate of saving. While the optimal, rational individual response to mortality risk in a frictionless setting is to pool that risk (Yaari, 1965; Davidoff et al., 2005; Reichling and Smetters, 2015), the corresponding response to longevity risk is less evident. Individuals could either bear it under a collective arrangement, or offload it at a cost by purchasing an annuity contract from an equity-backed insurance company. Both options allow individuals to pool mortality risk, but entail different implications with regard to longevity risk. We compare these arrangements to ascertain the option that maximizes individuals' expected utility. We also investigate the viability of the annuity contracts market by measuring the risk-return tradeoff with respect to longevity risk for the equity holders of the annuity contract provider.

Since the introduction of Group Self-Annuitization (GSA) by Piggott et al. (2005), retirement schemes in which individuals bear systematic risks as a collective, but pool idiosyncratic ones have captured the attention of scholars. The main novelty of our work is to concurrently model individual preferences and the business of an equity-holder-backed annuity provider when longevity risk exists. Despite the equity holders' critical role in the provision of contracts, comparisons of the *GSA* and annuity contracts that include longevity risk (e.g., Denuit et al., 2011; Richter and Weber, 2011; Maurer et al., 2013; Qiao and Sherris, 2013) disregard this aspect.

In order to credibly offer insurance against a systematic risk, the annuity provider requires a buffer capital that is constituted from either equity contribution and/or from contract loading to absorb unexpected shocks.[3] Either of these sources of

[1] Longevity and mortality risks are also referred to as *macro*- and *micro*-longevity risks, respectively.

[2] In 1975, close to 70% of all U.S. retirement assets were in DB plans. In 2015, DB assets accounted for only 33% of total retirement assets. Over the same period, assets in DC plans and Individual Retirement Accounts (IRAs) grew from 20% to 59% (Investment Company Institute, 2016).

[3] It would be equivalent to consider debt issuance to raise capital, and any dividend policy other

capital has a cost. If the annuity provider solicits capital from equity holders, then it would have to compensate equity holders with a longevity risk premium. If the provider charges too high a loading, then individuals would prefer the *GSA* over the annuity contract (e.g., Hanewald et al., 2013; Boyle et al., 2015).[4] Therefore, the existence of an annuity contract market hinges on the provider's ability to set a contract price such that all stakeholders are willing to participate in the market.

Existing estimates on individuals' willingness to pay to insure against longevity risk are low. Individuals are willing to offer a premium of between 0.75% (Weale and van de Ven, 2016) to 1% (Maurer et al., 2013) for an annuity contract that insures them against longevity risk without default risk. In contrast, the capital buffer that the annuity provider would have to possess to restrain its default risk is much larger (e.g., about 18% of the contract's best estimate value to limit the default rate to 1% in Maurer et al., 2013). These estimates suggest that the annuity provider has little capacity to compose its buffer capital only from contract loading. Equity capital provision is thus necessary, contrary to the common assumption that the full amount of the annuity provider's buffer capital is composed of loading charged to the individuals, as adopted in Friedberg and Webb, 2007; Richter and Weber, 2011; Maurer et al., 2013; Boyle et al., 2015. We attempt to reconcile the gap between the maximum loading that individuals are willing to pay, and the minimum capital necessary to provide annuity contracts that individuals are willing to purchase, by introducing equity holders. While analyses that incorporate both policy and equity holders exist in insurance (e.g., Filipović et al., 2015; Chen and Hieber, 2016), they are unforeseen in the literature on the comparison of the *GSA* with annuity contracts, which focuses on policy holders only.

Consistent with the inchoate market for longevity-hedging instruments, we assume that the annuity provider has no particular advantage in bearing longevity risk.[5] Moreover, the annuity provider is required to maintain the value of its assets above the value of its liabilities–a plausible regulatory requirement for such a for-profit entity. In contrast to the literature on collective schemes, which largely focuses on inter-generational risk-sharing (e.g., issues concerning its fairness and sta-

than a one-off dividend payment to equity holders (i.e., any gains before the end of the investment horizon are re-invested). This is because the Miller-Modigliani propositions on the irrelevance of capital structure (Modigliani and Miller, 1958) and dividend policy (Miller and Modigliani, 1961) on the market value of firms hold in our setup, which excludes taxes, bankruptcy costs, agency costs, and asymmetric information.

[4]While allocating retirement wealth between the annuity contract and the collective scheme is conceptually appealing, for the feasibility of a collective scheme, individuals can select only one option in our setting (e.g., mandatory participation in a collective scheme averts adverse selection, achieves cost reduction, etc., Bovenberg et al., 2007).

[5]Insurance companies may in practice have a comparative advantage in bearing longevity risk, such as the synergy of product offerings in terms of risk-hedging (Tsai et al., 2010), or the potential of life insurance sales in hedging longevity risk (i.e., natural hedging) (Cox and Lin, 2007; Luciano et al., 2015).

bility with respect to the age groups, see Gollier, 2008; Cui et al., 2011; Beetsma et al., 2012; Chen et al., 2015, 2016), we focus instead on risk-sharing between the individuals and the annuity provider's equity holders within a generation.

We begin by assuming that the annuity provider composes its buffer entirely from equity capital. In return for their capital contribution, equity holders receive the annuity provider's terminal wealth as a lump sum dividend. Due to equity-capital-cushioning, the annuity contract provides retirement benefits that have a lower standard deviation across scenarios. However, as equity capital is finite, there is a positive (albeit small) probability that the annuity provider will default. We infer the maximum loading that individuals are willing to offer, and the equity holders' risk-adjusted investment return.

We find that individuals marginally prefer the collective scheme. The Certainty Equivalent Loading (CEL), i.e., the level of loading on the annuity contract at which individuals would derive the same expected utility under either option, is slightly negative (i.e., -0.35% to -0.052%; Table 3). Furthermore, exposure to longevity risk does not enhance the equity holders' risk-return tradeoff if the annuity provider sells zero-loading contracts, because it yields only half of the Sharpe ratio of an identical investment without exposure to longevity risk, as well as a negative Jensen's alpha (Table 4). Consequently, the market-provided annuity contract would not co-exist with the collective scheme. The implication of our results would be even stronger if there were frictional costs, e.g., financial distress, agency, regulatory capital, and double taxation costs, because the equity holders would require a higher financial return from the capital they provide.

To further comprehend the tradeoff that an individual faces when selecting a contract, we carry out sensitivity tests with respect to the individual's characteristics, longevity risk, and the annuity provider's default risk. Our inference is robust to the deferral period (Table 7), stock exposure (Table 8), and parameter uncertainty surrounding the longevity model's time trend (Table 11). Situations characterized by extremities can intensify individual preference for either contract in an intuitive manner. For instance, the annuity contract is attractive to highly risk-averse individuals because its retirement benefits are less volatile (Table 6). If the equity capital is halved, the annuity provider's default risk rises markedly, and the annuity contract becomes less desirable to individuals (Table 9).

Greater uncertainty surrounding longevity evolution could lead to preference for the annuity contract, on the condition that the contract provider restrains default risk by raising more equity capital. If, for example, the longevity model's time trend variance is doubled, risk-averse individuals are willing to pay as much as 3.2% in loading for the annuity contract, but only if the provider has no default risk (Table 13). Under an alternative longevity model, which exhibits wider variation of survival at older ages, risk-averse individuals prefer the collective scheme, but only if the provider's default risk is eliminated too (Table 15). Despite any positive loading that individuals offer, none of the cases that we analyze show that the level of loading is sufficient to compensate equity holders (Tables 13 and 15). This is because in situations of heightened longevity risk, the equity holders' dividend is also more volatile, which compromises the financial performance of longevity risk exposure. Thus, there is no compelling support for annuity contract provision when individuals could form a collective scheme.

We present our model in Section 2 and calibrate it in Section 3. We first discuss the Baseline Case results from the individual's perspective (Section 4), then from the equity holders' point of view (Section 5). Section 6 is devoted to sensitivity tests on the individuals' traits, stock exposure, the annuity provider's leverage ratio, as well as the longevity model's attributes. We conclude in Section 7.

2 Model Presentation

We devise a model to investigate the welfare of individuals under a collective retirement scheme and a market-provided deferred variable annuity contract. The setting comprises a financial market with a constant risk-free rate and stochastic stock index, homogenous individuals with stochastic life expectancies, and two financial contracts for retirement.[6] We define and discuss these elements in detail in this section.

2.1 Financial Market

In a continuous-time financial market, the investor is assumed to be able to invest in a money market account and a risky stock index. The financial market is incomplete due to the lack of longevity-linked securities. We assume that annual returns to the risk-free asset are constant, r. The money market account is fully invested in the risk-free asset.

The value of the stock index at time t, which is denoted by S_t, follows the diffusion process, $dS_t = S_t\,(r + \lambda_S \sigma_S)\,\mathrm{d}t + S_t \sigma_S\,\mathrm{d}Z_{S,t}$. Z_S is a standard Brownian motion with respect to the physical probability measure, σ_S is the instantaneous stock price volatility, and $\lambda_S \sigma_S$ is the constant stock risk premium.

2.2 Individuals

At time $t_0 = 0$, individuals who are aged $x = 25$ either form a collective pension scheme or purchase a deferred annuity contract with a lump sum capital that

[6]We abstract from model uncertainty by assuming that the stochastic dynamics underlying the financial assets and life expectancies are known.

is normalized to one. Both retirement contracts commence retirement benefit payments at age 66, up to the maximum age of 95, conditional on the contract holders' survival. Individuals' lifespan is determined by survival probabilities that are modeled by the Lee and Carter (1992) model presented in Section 2.2.1.

2.2.1 Life Expectancy

We assume that individual mortality rates evolve independently from the financial market. Although productive capital falls as the population ages, empirical evidence on the link between demographic structure and asset prices is mixed (Erb et al., 1994; Poterba, 2001; Ang and Maddaloni, 2003; Arnott and Chaves, 2012).

We adopt the Lee and Carter (1992) model, which is widely used (e.g., by the U.S. Census Bureau and the U.S. Social Security Administration) and studied. This is a one-factor statistical model for long-run forecasts of age-specific mortality rates. It relies on time-series methods and is fitted to historical data. By relying on population mortality data, we eschew adverse selection that plagues the annuity market, i.e., the individuals who purchase an annuity typically have a longer average lifespan than the general population (Mitchell and McCarthy, 2002; Finkelstein and Poterba, 2004).

The log central death rate for an individual of age x in year t, $\log(m_{x,t})$[7] is assumed to linearly depend on an age-specific constant, and an unobserved period-specific intensity index, k_t:

$$\log(m_{x,t}) = a_x + b_x k_t + \varepsilon_{x,t} \qquad (1)$$

$\exp(a_x)$ is the general shape of the mortality schedule across age; b_x is the rate of change of the log central death rates in response to changes in k_t, whereas the error term, $\varepsilon_{x,t}$, is normally distributed with zero mean and variance σ_ε^2.

The Lee and Carter (1992) model is defined for the central death rates. By an approximation, we apply it to model the annual rate of mortality: letting $q_{x,t}$ be the probability that an individual of age x who is alive at the start of year t, dying before year $t+1$, $q_{x,t} \simeq 1-\exp(-m_{x,t})$. The probability that someone who is aged x at time t_0 is alive in s-year time, ${}_sp_x$, is then ${}_sp_x = \Pi_{l=0}^{s-1}(1-q_{x+l,t+l})$. We denote the conditional probability in year $t \geq t_0$ that an individual of age x at time t will survive for at least s more years as ${}_sp_x^{(t)}$, ${}_sp_x^{(t)} = \Pi_{l=0}^{s-1}(1-q_{x+l,t}) = \exp\left(\sum_{l=0}^{s-1} -m_{x+l,t}\right)$.[8]

[7] $m_{x,t}$ is the ratio of $D_{x,t}$, the number of deaths of an individual aged x in year t, over $E_{x,t}$, the exposure, defined as the number of aged x individuals who were living in year t. $m_{x,t} = \frac{D_{x,t}}{E_{x,t}}$.

[8] This is an exponentiated finite sum of log-normal random variables that has no known analytical distribution function. Therefore, we resort to simulation for our analysis. Alternate ways to proceed

While many refinements of Lee and Carter (1992) exist (e.g., the two-factor model of Cairns et al., 2006, the addition of cohort effects in Renshaw and Haberman, 2006), the model is not only reasonably robust to the historical data used, but also produces plausible forecasts that are similar to those from extensions of the model (Cairns et al., 2011).

2.2.2 Welfare

Individuals maximize expected utility in retirement.[9] At this time, benefits from the retirement contracts constitute the individual's only source of income for the individuals. We consider individuals who exhibit Constant Relative Risk Aversion (CRRA), and evaluate their utility in retirement by Equation (2).

$$U(\Xi) = \int_{t_R}^{T} \mathrm{e}^{-\beta t} \frac{\Xi_t^{1-\gamma}}{1-\gamma} {}_{t-t_0}p_{25}\, \mathrm{d}t \tag{2}$$

${}_{t-t_0}p_{25}$	=	probability that someone who is 25 years old in year t_0 is alive in year t
β	=	subjective discount factor
γ	=	risk aversion parameter, $\gamma > 1$
Ξ_t	=	retirement income in year t
t_R	=	retirement year, i.e., $t_R = t_0 + 66 - 25$
T	=	year of maximum age, i.e., $T = t_0 + 95 - 25$

2.3 Financial Contracts for Retirement

There are two financial contracts for retirement. The first is a collective pension called the Group Self-Annuitization (*GSA*) scheme. The second is a Deferred Variable Annuity (*DVA*) contract offered by an annuity provider who is backed by equity holders. We describe both contracts in this section. Appendix A elaborates on the rationale of the definition and provision of the contracts.

The financial contracts specify the distribution of financial and longevity risks among the stakeholders. As the contracts are intended to underscore longevity risk, both treat stock market risk identically - the risk is fully borne by the contract holders. The benefits due, henceforth known as entitlements, are fully indexed to the same underlying financial portfolio called the reference portfolio (e.g., a portfolio that is 20% invested in the stock index, and 80% in the money market account).

include estimating the quantiles of the random survival probabilities (e.g., Denuit et al., 2011), or the Taylor series approximation by Dowd et al. (2011).

[9] We can ignore bequest motives as both contracts provide income only when the individual is still alive.

Thus, if the *DVA* provider adopts the reference portfolio's investment policy, the provider is hedged against financial market risk.

Longevity risk distribution, however, distinguishes the two contracts. Under the *GSA*, it is shared equally among individuals. Under the *DVA*, the risk is borne by the equity holders up to a limit implied by their equity contribution, beyond which the *DVA* provider defaults. Both contracts stipulate to distribute mortality credit according to the survival probabilities, conditional on the date of contract sale. The *DVA* provider (that is, its equity holders) bears the risk that the survival probability forecast deviates from their realized values, when the provider is required to either dip into its equity capital to finance underestimation of longevity, or to collect the excess arising from overestimation of longevity. After the final payment is made, the provider disburses any surplus to equity holders as a dividend.

Due to the non-existence of financial assets that are associated with longevity risk, the risk cannot be hedged by the *DVA* provider. Additionally, we assume that the number of individuals who either purchase the *DVA* or participate in a *GSA* is large enough such that by the Law of Large Numbers, the proportion of surviving individuals within each pool coincides with that implied by the realized survival probabilities, so we can eliminate mortality risk.[10]

2.3.1 Deferred Variable Annuity (*DVA*)

The *DVA* contract is parametrized by an actuarial construct called the Assumed Interest Rate (*AIR*), $h := \{h_t\}_{t=t_0}^{T}$. The *AIR* is a deterministic rate that determines the cost, A, of a contract sold to an individual who is aged x at time t_0 as follows:

$$A(h, F, t_0, x) := (1+F) \int_{t=t_R}^{T} {}_{t-t_0}p_x^{(t_0)} \exp\left(-h_t \times (t - t_R)\right) \mathrm{d}t \tag{3}$$

$$\begin{aligned} {}_{t-t_0}p_x^{(t_0)} &= \text{conditional probability in year } t_0 \\ &\quad \text{that a individual of age } x \text{ lives} \\ &\quad \text{for at least } t-t_0 \text{ more years} \\ h &= AIR \\ F &= \text{loading factor} \\ t_R &= \text{retirement year, i.e., } t_R = t_0 + 66 - 25 \end{aligned}$$

[10]The *GSA* in our setting is a specific case of the *GSA* of Piggott et al. (2005) because by omitting mortality risk and assuming an identical investment portfolios for every member, the pooling of idiosyncratic risks–a defining feature of the *GSA*– is irrelevant.

The loading factor, F, is a proportional one-off premium that the *DVA* provider attaches to a contract. A contract that is priced at its best estimate has a loading factor of zero, $F = 0$.

The *DVA* contract is indexed to a reference investment portfolio that follows a deterministic investment policy, $\theta := \{\theta_t\}_{t=t_0}^{T}$. θ_t is the fraction of portfolio wealth allocated to the risky stock index at time t, while the remaining $1-\theta_t$ is invested in the money market account. Let $W_t^{Ref}(\theta)$ be the value of the reference portfolio at time t. The dynamics of the reference portfolio are thus $\mathrm{d}W_t^{Ref} = W_t^{Ref}(r+\theta_t\lambda_S\sigma_S)\,\mathrm{d}t + W_t^{Ref}\theta_t\sigma_S\,\mathrm{d}Z_{S,t}$. Using an annuitization capital that is normalized to one, the individual purchases $A(h^*, F, t_0, x)^{-1}$ unit(s) of *DVA* contract(s), and is entitled to Ξ, for every year t in retirement, $t_R \leq t \leq T$.[11]

$$
\begin{aligned}
\Xi(h, F, t, x) &:= \frac{\exp\left(-h_t \times (t - t_R)\right)}{A(h, F, t_0, x)} \times \frac{W_t^{Ref}(\theta)}{W_{t_0}^{Ref}(\theta)} \qquad (4)\\
W_t^{Ref}(\theta) &= \text{value of the reference portfolio at time } t
\end{aligned}
$$

The *AIR* influences the expectation and dispersion of the benefit payments over time. For instance, the fund units are front- (back-) loaded (i.e., due in the earlier (later) years of retirement) under a higher (lower) *AIR*.[12]

We demonstrate in Appendix A that for any given θ, the *AIR* that maximizes the individual's expected utility in retirement is Equation (5), which we refer to as the optimal *AIR*, h^*. h^* depends on the individual's preference and financial market parameters. It serves as the *AIR* of both the *DVA* and *GSA*.

$$
\begin{aligned}
h^*(t, \theta_t) &:= r + \frac{\beta - r}{\gamma} - \frac{1-\gamma}{\gamma}\theta_t\sigma_S\left(\lambda_S - \frac{\gamma\theta_t\sigma_S}{2}\right) \qquad (5)\\
t &= \text{time index}, t, t_R \leq t \leq T\\
r &= \text{constant short rate}\\
\beta &= \text{subjective discount factor}\\
\gamma &= \text{risk aversion parameter}\\
\theta_t &= \text{fraction of wealth allocated to the stock index}\\
\sigma_S &= \text{diffusion term of the stock index}\\
\lambda_S &= \text{Sharpe ratio of the stock index}
\end{aligned}
$$

[11] The benefits adjust instantaneously with the value of the portfolio to which the contract is indexed. Maurer et al. (2014) make the case for smoothing of the benefits, which is advantageous to both the policyholder and the contract provider.

[12] Let $\tilde{r}$ denote the reference portfolio's expected return, and suppose h is time-invariant. Then an annuity contract with $h = \tilde{r}$ has a constant expected benefit payment path. When $h < \tilde{r}$, then the expected benefit stream is upward sloping, with increasing variance as the individual ages. Conversely, when $h > \tilde{r}$, the expected benefit stream is downward sloping, and the variance is higher during the initial payout phase. Horneff et al. (2010) provides an exposition on retirement benefits under numerous *AIR*s and reference portfolios.

The *DVA* provider merely serves as a distribution platform for annuity contracts. It acts in the best interest of its equity holders, who are assumed to outlive the individuals. The equity holders provide a lump sum capital that is proportional to the value of its estimated liabilities in the year t_0.[13] At every date $t \geq t_0$, the *DVA* provider's asset value has to be at least equal to the value of its estimated liabilities. In any year $t_0 \leq t \leq T$, if the *DVA* provider fails to meet the 100% solvency requirement, then the *DVA* provider defaults. Regulatory oversight is introduced for the *DVA* provider, because as a for-profit entity, the *DVA* provider may have an incentive to take excessive risk at the individuals' expense (Filipović et al., 2015), by adopting a high leverage ratio, for example. The individual receives a benefit that is equal to the *DVA* entitlement,

$$\Xi^{DVA}(h^*, F, t, x) \quad = \quad \Xi(h^*, F, t, x) \tag{6}$$

in every year of retirement, conditional on the individual's survival and the *DVA* provider's solvency. $\Xi(.)$ is Equation (4) while h^* is Equation (5).

In the event of default, the residual wealth of the *DVA* provider is distributed among all living individuals, in proportion to the value of their contracts that remains unfulfilled. Equity holders receive none of the residual wealth. We impose a resolution mechanism that obliges individuals to use the provider's liquidated wealth to purchase an equally-weighted portfolio of zero-coupon bonds, of maturities from the year of default if the individual is already retired, or from the year of retirement, until the year of maximum age. Assuming that the bond issuer poses no default risk, then the individual has a guaranteed income until death, but receives no mortality credit. If the individual dies before the maximum age, the face value of the bonds that mature subsequently is not bequeathed. This resolution to insolvency is harsh on the individuals because it eliminates the mortality credit, but it reflects the empirical evidence that individuals substantially discount the value of an annuity that poses default risk (Wakker et al., 1997; Zimmer et al., 2009).

2.3.2 Group Self-Annuitization (GSA)

Similar to the *DVA*, the *GSA*'s entitlement is parameterized by the optimal *AIR*, h^*, and is indexed to a reference portfolio with the investment policy θ. The aged-x individual receives $A\left(h^*, 0, t, x\right)^{-1}$ contract(s) for every unit of contribution at time t. In any year $t \geq t_R$, the *GSA*'s entitlement depends on the reference portfolio's value at time t, $W_t^{Ref}(\theta)$.

The description of the *GSA* thus far is identical to a *DVA* contract with zero loading, $F = 0$. The *GSA*'s distinctive feature is that the entitlements are adjusted according to its funding status. Let the funding ratio at time t, FR_t, be the ratio of the *GSA*'s value of assets, taking into account the investment return from the

[13]The estimation of the value of liabilities is explained in Appendix B.

preceding year, over the best estimated value of its liabilities.[14] For any year t in retirement, $t_R \leq t \leq T$, the contract holder is entitled to $\Xi^{GSA}(h^*, 0, t, x)$.

$$\begin{aligned} \Xi^{GSA}(h^*, 0, t, x) &= \Xi(h^*, 0, t, x) \times \frac{FR_t}{1} \\ &= \frac{\exp\left(-h^*(t, \theta_t) \times (t - t_R)\right)}{A(h^*, 0, t_0, x)} \times \frac{W_t^{Ref}(\theta)}{W_{t_0}^{Ref}(\theta)} \times FR_t \\ FR_t &= \text{Funding Ratio in year } t \end{aligned} \tag{7}$$

The first two terms of Equation (7) are identical to the entitlement for a *DVA* contract with zero loading, Equation (4). The final term of Equation (7) represents the adjustment. If FR_t is smaller (larger) than 1, then the *GSA* entitlement, Ξ^{GSA}, is lower (higher) than the *DVA* entitlement, Ξ^{DVA}, in year t. Equation (7) ensures that the *GSA* is 100% funded in any year.

3 Model Calibration

We consider three groups of individuals, distinguished by their risk aversion levels, $\gamma = 2, 5$, and 8.[15] Individuals are otherwise homogenous. They have an annual subjective discount factor of 3%,[16] are aged 25 at time $t_0 = 0$, and use a lump sum that is normalized to one, to either purchase *DVA*(s), or to join the *GSA* at time t_0. Both contracts stipulate payment of annual retirement benefits from age 66 until age 95, conditional on the individual's survival in any year, according to the contract specification in Section 2.3.

The portfolio to which the *DVA* and *GSA* are indexed is either fully invested in the money market account ($\theta = 0$), or 20% invested in equities and 80% in the money market account ($\theta = 20\%$). These allocations yield the optimal *AIR* range of 3-4% (Table 1) that is not only observed in the annuity market (Brown et al., 2001), but also typically considered in the related literature (Koijen et al., 2011; Maurer et al., 2013). In Section 6.3, we explore alternative investment policies and demonstrate that they uphold the same results as when $\theta = 0, 20\%$.

We assume that the *DVA* provider's equity holders provide a lump sum capital at date t_0 that is 10% of the contract's best estimate price. The level of equity capital contribution is set such that the annuity provider's leverage ratio (i.e.,

[14] Estimation of the *GSA* liabilities is identical to the estimation of liabilities of the *DVA* provider. See Appendix B for details.

[15] Using survey responses from the Health and Retirement Study on the U.S. population, Kimball et al. (2008) estimate that the mean risk aversion level among individuals is 8.2, with a standard deviation of 6.8.

[16] While field experiments reveal a wide range of implied subjective discount factor (e.g., see Table 1 in Frederick et al., 2002), we choose a value that is commonly adopted in welfare analysis. For example, in similar analyses on retirement income, Feldstein and Ranguelova (2001) and Hanewald et al. (2013) adopt a subjective discount factor of around 2%.

Leverage Ratio := 1 − Value of Equity/Value of Assets) is 90%. This reflects the average of the leverage ratio of U.S. life insurers between 1998-2011.[17]

To provide descriptive calculations on individual welfare under the *GSA* and the *DVA*, we calibrate the financial market and life expectancy models to U.S. data. These parameters constitute our Baseline Case.

3.1 Financial Market

We adopt a constant interest rate of $r = 3.6\%$. The stock index has an annualized standard deviation of $\sigma_S = 15.8\%$, and an instantaneous Sharpe ratio of $\lambda_S = 0.467$. This implies that the stock risk premium is $\lambda_S \sigma_S = 7.39\%$. These parameters reflect the performance of the market-capitalization-weighted index of U.S. stocks and the yield on the three-month U.S. Treasury bill over the recent past.

3.2 Life Expectancy

We estimate the Lee and Carter (1992) model using U.S. female death counts, $D_{x,t}$, and the population's exposure to risk, $E_{x,t}$, from 1980 to 2013 from the Human Mortality Database.[18] The mortality rate for age group x in year t is thus $D_{x,t}/E_{x,t}$.

Estimation of the Lee and Carter (1992) model proceeds in three steps. First, k_t is estimated using Singular Value Decomposition. In the second step, a_x and b_x are estimated by Ordinary Least Squares on each age group, x. In the third step, k_t is re-estimated by iterative search to ensure that the predicted number of deaths coincides with the data. For identification of the model, we impose the constraints $\sum_x b_x = 1$ and $\sum_t k_t = 0$.

The estimated model is used for forecasting by assuming an $ARIMA(0,1,0)$ time series model for the mortality index k_t.

$$
\begin{aligned}
k_t &= c + k_{t-1} + \delta_t \\
& \quad \delta \sim \mathcal{N}\left(0, \sigma_\delta^2\right)
\end{aligned}
\tag{8}
$$

[17] Based on the A.M. Best data used in Koijen and Yogo (2015), the leverage ratio of U.S. life insurers between 1998 to 2011 is 91.36% on average. Assuming that assets are composed of premium and equity capital only, and normalizing Premium = 1, we have Leverage Ratio = $1 - \text{Equity}/(1+\text{Equity})$, which we use to solve for Equity when the Leverage Ratio ≈ 90%.

[18] This fitting period is selected using the method of Booth et al. (2002). It involves defining fitting periods starting from the first year of data availability till the last year of data availability, and progressively increasing the starting year. A ratio of the mean deviance of fit of the Lee and Carter (1992) model with the overall linear fit is computed for these fitting periods. The period for which this ratio is substantially smaller than that for periods starting in previous years is chosen as the best fitting period.

Forecasts of the log of the central death rates for any year t', $t' \geq t$, are given by $\mathbb{E}_t\left[\log\left(m_{x,t'}\right)\right] = a_x + b_x\hat{k}_{t'}$, with $\hat{k}_{t'} = (t'-t)\,c + k_t$. The realized log of the mortality rate incorporates the independently and identically normally distributed error terms $\varepsilon_x \sim \mathcal{N}\left(0, \sigma_x^2\right)$ and $\delta \sim \mathcal{N}\left(0, \sigma_\delta^2\right)$, with $\operatorname{cov}\left(\varepsilon_{x,t_1}, \delta_{t_2}\right) = 0$ for any $t_1, t_2 \in [t_0, T]$ and x. Therefore, the conditional expected forecast error of $\log\left(m_{x,t}\right)$ is zero.

We estimate that $\hat{c} = -1.0689$, which implies a downward trend for k_t, while the estimate of σ_δ is $\widehat{\sigma_\delta} = 1.781$. In Figure 1, we present the estimates for a_x, b_x, and σ_x. From age 10 onwards, a_x is increasing in age. Estimates for b_x suggest that the change in the sensitivity of age groups to the time trend, k, is not monotone across ages. As for σ_x, it decreases in age non-monotonically until around age 85. With these estimates, 83.8% of the variation in the data is explained.

In Figure 2, we display a fan plot of the fraction of living individuals by age, between 25 and 95, with the population at age 25 normalized to one. The plot implies that the fraction of living individuals in retirement can vary over a wide range (i.e., the difference between the maximum and minimum realization). At its peak at age 88, the range of the proportion of living individuals is as wide as 30%.

Figure 1: Lee and Carter (1992) Parameter Estimates

The top panel shows the estimates for a_x, the middle panel displays the estimates for b_x, whereas the bottom panel presents the estimates of σ_x, for the Lee and Carter (1992) model as specified by Equation (1). The calibration sample is the U.S. Female Mortality data from 1980 to 2013, from the Human Mortality Database. The estimate of c is -1.0689 and that of σ_δ is 1.781. 83.8% of variation of the sample is explained by these estimates.

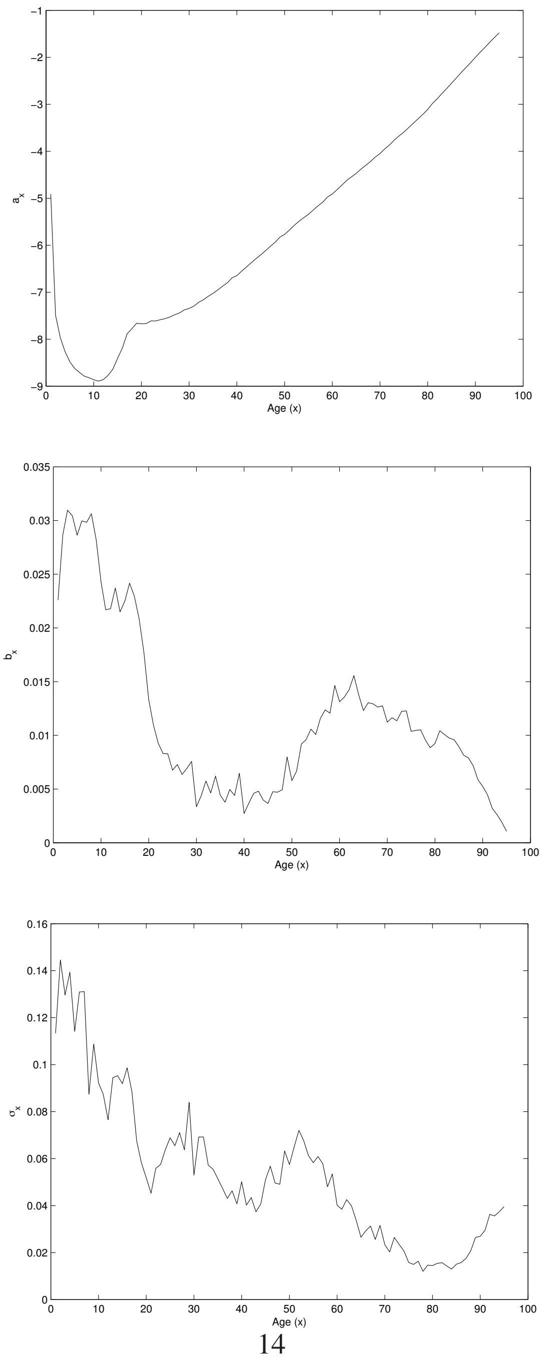
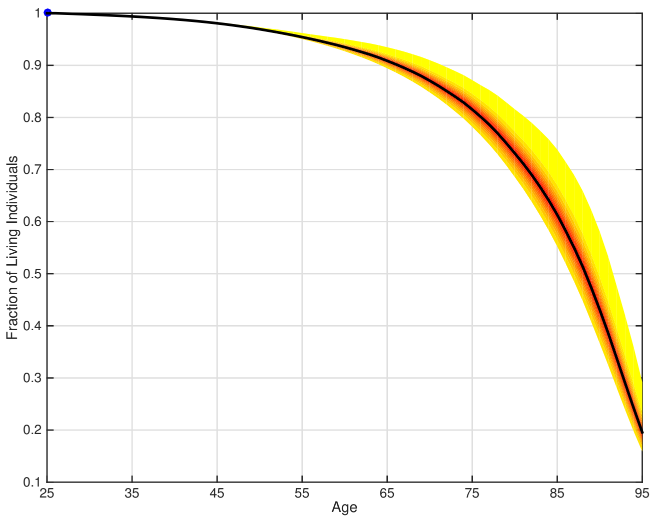


Figure 2: Lee and Carter (1992) Fan Plot
This figure presents the fan plot of the simulated fraction of living individuals (i.e., the population of 25-year-olds is normalized to one) over 10,000 replications when longevity is modeled according to Lee and Carter (1992), using estimates in Figure 1. Darker areas indicate higher probability mass.



3.3 Contract Characteristics

In order to develop intuition and grasp the contracts' definition, we discuss the characteristics of the *GSA* and the *DVA* under the calibrated parameters. Table 1 presents the optimal *AIR*s as given by Equation (5), and evaluated at the parameters outlined in Sections 3.1 and 3.2. The optimal *AIR*s have a range of 3-4% which is common in the annuity market (Brown et al., 2001) and in the related literature (Koijen et al., 2011; Maurer et al., 2013).

Table 1: Baseline Case: Optimal *AIR*, h^* (%)
This table shows the optimal *AIR*, Equation (5), of the *DVA* and *GSA* contracts by the individuals' risk aversion parameter, γ. The underlying portfolio to which the contracts are indexed is either 100% invested in the money market account ($\theta = 0$), or 20% in the risky stock index and 80% in the money market account ($\theta = 20\%$).

θ	γ		
(%)	**2**	**5**	**8**
0	3.31	3.50	3.54
20	4.00	4.48	4.48

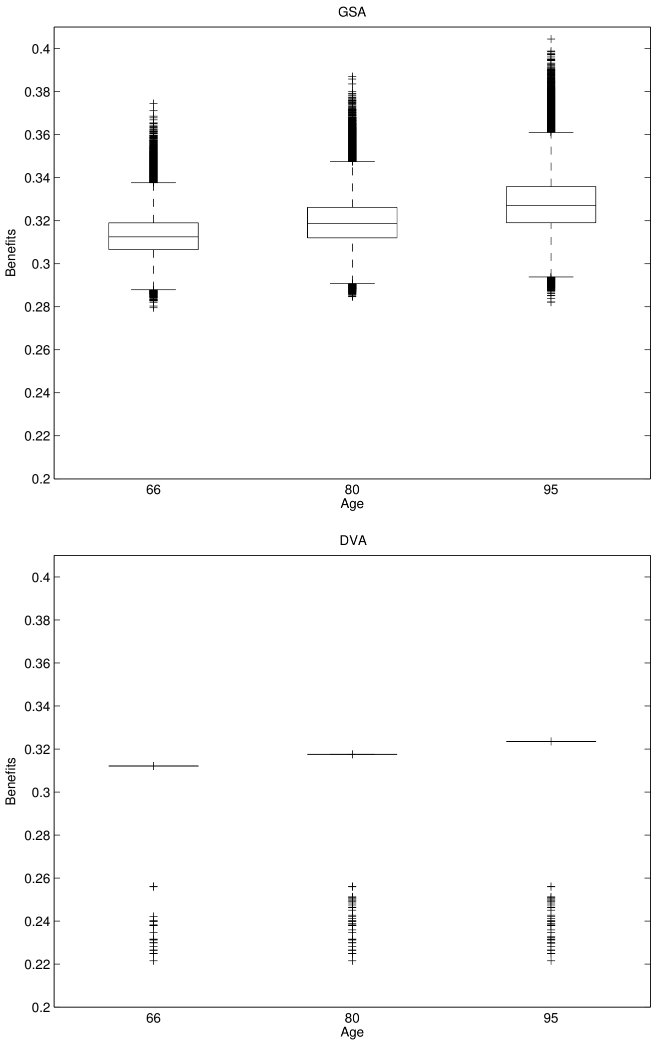
Figure 3 is a box plot of the benefits that individuals receive under the *DVA* and the *GSA*. The median benefits of both contracts grow along the retirement horizon due to larger mortality credits at higher ages. For the *DVA*, the median value is also the maximum, because the surplus from life expectancy misestimates belongs to the equity holders, not to the individuals.

The *GSA* yields more instances of positive than negative adjustments to benefits that are 1.5-time larger than the range between its 75^{th} and 25^{th} percentiles. We infer this from the relative density of "+" symbols above and under the box (Figure 3, top panel). When the individual attains the maximum age of 95, benefits as large as 25% more than the median could occur. In contrast, in the worse scenario at the same age, the reduction in benefits relative to the median is, at most around 13% only, at most. This asymmetric effect on benefits arises from the non-linearity of the Lee and Carter (1992) model. For error terms of the same magnitude (i.e., $\{\varepsilon_{x,t}\}_{t=t_0}^{T}$ in Equation (1) and $\{\delta_t\}_{t=t_0}^{T}$ in Equation (8), for any $x \in \mathbb{Z} \cap [25, 95]$), overestimation of the log of the central death rates generates a larger entitlement adjustment than underestimation does. Besides, when the *DVA* provider defaults, the individual is at risk of receiving a much lower benefit. The worst case under the *GSA* entails up to a 30% lower benefit relative to the median at the maximum age.

The box plots indicate that while both contracts offer comparable benefits at the median, those of the *GSA* have higher standard deviations across scenarios due to the entitlement adjustments, but upward adjustments are more prevalent than downward ones. The *DVA* offers less volatile benefits, but is susceptible to severe low benefit outcomes when the provider defaults. These are the main features that the individuals weigh in utility terms.

Figure 3: Box Plots of *GSA* and *DVA* Benefits

The figure presents the box plot of benefits, for the *GSA* (top panel), and the *DVA* (bottom panel), for an individual with a risk aversion level of $\gamma = 5$, at ages 66, 80 and 95. The underlying portfolio is invested in the money market account only. The line in the middle of the box is the median, while the edges of the box represent the 25^{th} and 75^{th} percentiles. The height of the box is the interquartile range, i.e., the interval between the 25^{th} and 75^{th} percentiles. The "+" symbols represent data points that are 1.5 times larger than the interquartile range.



4 The Individual's Perspective

We investigate two settings distinguished by the existence of stock market risk. In both, there is longevity risk, but in one instance, there is no investment in the stock market, $\theta = 0$, and so the financial return is constant at r, whereas in the other, $\theta = 20\%$ is invested in the risky stock index while the remaining 80% is allocated to the money market account. All results are based on simulations with 500,000 replications unless specified otherwise. The code that produces all figures and estimates in Sections 4 and 6 are available from the authors upon request.

4.1 Cumulative Default Rate

We measure the *GSA* provider's default rates with the Cumulative Default Rate, an estimate of the probability that the *DVA* provider defaults during the individuals' planning horizon.

Let D_t be the indicator function that the *DVA* provider has defaulted in any year t', $t_0 < t' \leq t \leq T$. For example, if the *DVA* provider defaults in the year t^*, then $D_t = 1$ for $t \geq t^*$ and $D_t = 0$ for $t < t^*$. Additionally, $D_{t_0} \equiv 0$ because the contracts are sold at their best estimate price, and the equity contribution is non-negative.

The marginal default rate in year t, $d(t)$ is the probability that the annuity provider defaults in year t, conditional on not having defaulted in previous years.

$$\begin{aligned} d(t) &:= \text{Marginal Default Rate in year } t \\ &= \frac{\mathbb{E}[D_t]}{1 - \mathbb{E}[D_t]} \end{aligned} \tag{9}$$

We define the Cumulative Default Rate as

$$\begin{aligned} \text{Cumulative Default Rate} &:= 1 - \Pi_{t=t_0}^{T} (1 - d(t)) \\ d(t) &= \text{Equation (9)} \end{aligned} \tag{10}$$

The default rates in the Baseline Case are at most 0.01% (Table 2). As the *AIR* determines whether the bulk of benefits are due earlier or later in retirement, when combined with the fact that longevity forecast errors are larger at longer horizons, the *DVA* provider's default rates are inversely related to the *AIR*s. A higher *AIR* results in a payment schedule with benefits mostly due earlier in retirement. As such, the longevity estimates are accurate when most of the benefits are paid. Conversely, if the *AIR* is low, benefit payments are deferred to the end of retirement, when life expectancies are most vulnerable to forecasting errors. Therefore, for a fixed level of equity capital, the *DVA* provider is less susceptible to defaults when the *AIR* is higher.[19] For the risk aversion levels $\gamma = 2, 5, 8$, the optimal *AIR* is

[19]From the regulator's perspective, the notion of an annual probability of default, instead of a

increasing in γ (Table 1), hence the default rates are decreasing in γ (Table 2) for both $\theta = 0, 20\%$. Similarly, the default rates are lower when $\theta = 20\%$ than when $\theta = 0\%$ for all levels of γ because the optimal *AIR*s are higher under $\theta = 20\%$.

Table 2: Baseline Case: Cumulative Default Rates (%)

This table displays the Cumulative Default Rates, Equation (10), of the *DVA* provider who sells zero-loading variable annuity contracts with a 40-year deferral period, and has equity capital valued at 10% of the liabilities in the year that the contract was sold. The underlying portfolio to which the *DVA* and *GSA* are indexed is either fully invested in the money market account ($\theta = 0$), or 20% in the stock index, and 80% in the money market account ($\theta = 20\%$).

θ	γ		
(%)	**2**	**5**	**8**
0	0.0102	0.0084	0.0082
20	0.0070	0.0038	0.0038

4.2 Individual Preference for Contracts

We quantify the individuals' preference for the contracts via the Certainty Equivalent Loading (*CEL*). This is the level of loading on the *DVA* (i.e., F in Equation (3)), that equates an individual's expected utility under the *DVA* and the *GSA*, i.e., the value such that Equation (11) holds. A positive (negative) *CEL* suggests that the individual prefers the *DVA* (*GSA*).

$$\begin{aligned}
\mathbb{E}\left[U\left(\frac{1}{1+CEL}\times \Xi^{DVA}|_{F=0}\right)\right] &= \mathbb{E}\left[U\left(\Xi^{GSA}\right)\right] \qquad (11)\\
\Xi^{DVA}|_{F=0} &= \text{Retirement benefits, Equation (6),}\\
&\quad\ \text{of a } DVA \text{ with zero loading, } F=0\\
\Xi^{GSA} &= \text{Retirement benefits, Equation (7),}\\
&\quad\ \text{of a } GSA\\
U(.) &= \text{Utility function, Equation (2),}
\end{aligned}$$

Confidence intervals for the *CEL*s are estimated via the Delta Method, for which more details are in Appendix C.

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cumulative one may be more salient. We explore the "Maximum Annual Conditional Probability of Default", defined as $\max_{\{t=t_0,\ldots,T\}} d(t)$, and find that the maximum annual default rate in the Baseline Case is 0.0008%. This suggests that the 10% buffer capital is sufficient to restrict default rates of *DVA* providers who are exposed to only longevity risk to existing regulatory limits (e.g., Solvency II for insurers in Europe).

Table 3 presents the *CEL* in the Baseline Case. The *CEL*s are negative for all risk aversion levels. This implies that individuals prefer the *GSA* over the *DVA*, but only marginally. If the *DVA* contracts were to be sold at a discount of between 0.052% and 0.350%, then individuals would be indifferent between the two contracts. Besides, the *CEL* is increasing in the risk aversion level, γ. This is because more risk-averse individuals have a greater preference for the *DVA* benefits' lower standard deviation across scenarios.

Table 3: Baseline Case: Certainty Equivalent Loading (CEL) (%)
This table presents the *CEL*, Equation (11), by the risk aversion levels (γ). Individuals aged 25 purchase either the *DVA* or join the *GSA* with a lump sum capital normalized to one. The reference portfolio is either fully invested in the money market account ($\theta = 0$), or is $\theta = 20\%$ invested in the stock index and 80% in the money market account. The expected utilities to which the *CEL*s are associated are computed over individuals' retirement between ages 66 and 95. The equity holder's capital is 10% of the present value of liabilities at the date when the contract is sold. The default rates that ensue at this level of equity capitalization are shown in Table 2. The 99% confidence intervals estimated by the Delta Method are in parentheses.

θ	γ		
(%)	**2**	**5**	**8**
0	-0.350 [-0.362, -0.339]	-0.200 [-0.211, -0.188]	-0.055 [-0.067, -0.044]
20	-0.349 [-0.361, -0.338]	-0.200 [-0.216, -0.184]	-0.052 [-0.088, -0.016]

5 The Equity Holders' Perspective

To evaluate the equity holders' risk-return tradeoff on longevity risk exposure, we consider two widely used measures of performance: the Sharpe ratio and the Jensen's alpha, of providing capital to the annuity provider, against those of investing the same amount of capital in the reference portfolio over the same time period.[20] As in Section 4, the annuity provider offers contracts at zero loading.

[20]The stochastic discount factor, $\{M_t\}_{t=t_0}^{T}$, that follows $\frac{dM_t}{M_t} = -r\,dt - \lambda_S\,dZ_{S,t}$, allows us to price any contingent claim exposed to stock market risk only: If X_t is a (random) cash flow generated by a contingent claim at time t, then its price at time t_0 is $\mathbb{E}_{t_0}\left[\int_{t=t_0}^{T} \frac{M_t}{M_{t_0}} X_t\,dt\right]$. However, when such pricing is carried out for claims due on a long horizon, and the market price of stock risk (i.e., the Sharpe ratio) exceeds its volatility, the price depends on extreme sample paths along which the claim's return explodes (Martin, 2012). As the claims are susceptible to severe underpricing when the Monte Carlo replication sample size is small, we refrain from valuing contingent claims when comparing the equity holders' investment opportunities.

Equity holders contribute 10% of the best estimated value of the *DVA* provider's liabilities at time t_0, and receive the terminal wealth of the annuity provider, $W_T^{(A)}$, as a dividend. When the best estimated value of the *DVA* provider's liabilities is normalized to one, the continuously compounded annualized return of capital provision, in excess of the risk-free rate of return, is thus $R^{(A_{exs})} = \log\left(W_T^{(A)}/0.1\right)/(T-t_0)-r$. We evaluate the equity holders' profitability via the Sharpe ratio, $SR = \mathbb{E}\left[R^{(A_{exs})}\right]/\sigma^{(A_{exs})}$, and we compute the Sharpe ratio's confidence intervals in accordance with Mertens (2002).

The Jensen's alpha, α, is given by Equation (12) (Jensen, 1968).

$$R^{(A_{exs})} = \alpha + \beta R^{(S_{exs})} + u \tag{12}$$

$R^{(S_{exs})}$ is the annualized return of the stock index in excess of the return on the money market account, and u is the error term. We estimate Equation (12) by Ordinary Least Squares. α assesses the investment performance of providing capital to the annuity provider, relative to that of the market portfolio, on a risk-adjusted basis. A positive α suggests that longevity risk exposure enhances the equity holders' risk-return tradeoff. When $\theta = 0$, $\beta = 0$ due to the assumption that the mortality evolution is uncorrelated with the financial market dynamics. If in Equation (12), $R^{(A_{exs})}$ is replaced by the annualized return in excess of the risk-free rate of return for the reference portfolio, then $\alpha = 0$ and $\beta = \theta$. This is because the reference portfolio has identical financial market risk exposure as capital provision, but is not exposed to longevity risk.

When $\theta = 0$, the annualized excess return of capital provision is between -0.008 and -0.007% and the standard deviation is 3.9% (Table 4, top panel). Relative to the zero excess return from investing in the money market account, equity capital provision is inferior, but the difference is economically insignificant. When $\theta = 20\%$, investing in the *DVA* provider yields an expected excess return of 1.44% (Table 4, bottom panel). This is of no material difference with the expected excess return on the identical financial market portfolio, i.e., $\theta\lambda_S\sigma_S - \theta^2\sigma_S^2/2 = 1.43\%$ when $\theta = 20\%$. However, the standard deviation of excess returns is considerably higher when equity holders are exposed to longevity risk (i.e., $\approx 5\%$, Table 4, bottom panel), than when their investment is subject to stock market risk only (i.e., $\theta\sigma_S = 3.17\%$ with $\theta = 20\%$). Consequently, investing in the financial market only is associated with a Sharpe ratio around 50% higher than the Sharpe ratio of providing capital to the *DVA* provider (i.e., 0.29 in Table 4, bottom panel, as compared to $\lambda_S - \frac{\theta\sigma_S}{2} = 0.45$[21] when $\theta = 20\%$). Thus, if equity holders were risk-neutral, then the excess returns imply that they would be indifferent between either investment opportunity. If equity holders were risk averse, then by the Sharpe ratio, investing

[21] This is the discrete Sharpe ratio, which is the parameter we estimate using simulation replications, as opposed to the instantaneous Shape ratio, λ_S (Nielsen and Vassalou, 2004).

in longevity risk worsens the equity holders' risk-return tradeoff when the annuity provider sells the contracts at zero loading. This is corroborated by the negative Jensen's alpha of -0.0001. Yet, even at zero loading, individuals prefer the *GSA* over the *DVA*. Any positive loading is infeasible, because it would only intensify individuals' preference for the *GSA*. Therefore, the annuity provider is incapable of adequately compensating its equity holders for exposure to longevity risk.

Table 4: Baseline Case: Equity Holders' Investment Performance Statistics
This table displays the equity holders' mean annualized return in excess of the risk-free rate of return ($\mathbb{E}\left[R^{(A_{exs})}\right]$, %), standard deviation of annualized excess return ($\sigma^{(A_{exs})}$, %), the Sharpe ratio (*SR*) and Jensen's alpha ($\mathbb{E}[\alpha]$, %), Equation (12), of capital provision to the *DVA* provider. The underlying portfolio is either invested in the money market account only ($\theta = 0$, top panel), or is 20% invested in the risky stock index, and 80% invested in the money market account ($\theta = 20\%$, bottom panel). The 99% confidence intervals are in parentheses.

$\theta = 0$

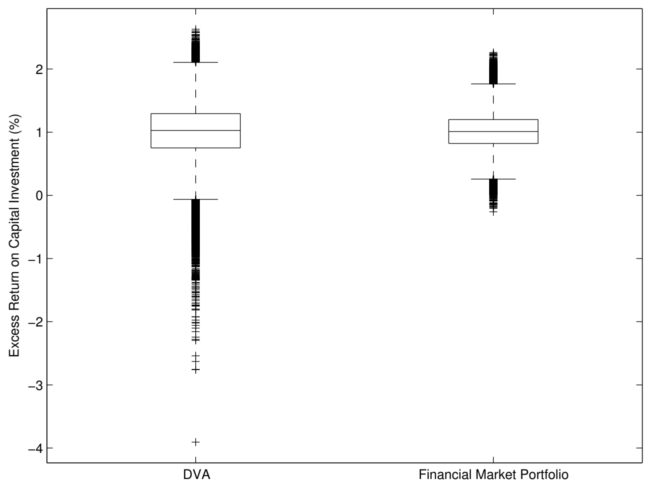
Statistic	γ		
	2	**5**	**8**
$\mathbb{E}\left[R^{(A_{exs})}\right]$ (%)	-0.008 [-0.010, -0.006]	-0.007 [-0.009, -0.005]	-0.007 [-0.008, -0.005]
$\sigma^{(A_{exs})}$ (%)	3.96 [3.95, 3.40]	3.91 [3.90, 3.91]	3.89 [3.88, 3.90]
SR	-0.002 [-0.0056, 0.0016]	-0.0017 [-0.0054, 0.0019]	-0.0017 [-0.0053, 0.0020]
$\mathbb{E}[\alpha]$ (%)	-0.0001 [-0.0001, -0.0001]	-0.0001 [-0.0001, -0.0001]	-0.0001 [-0.0001, -0.0001]

$\theta = 20\%$

Statistic	γ		
	2	**5**	**8**
$\mathbb{E}\left[R^{(A_{exs})}\right]$ (%)	1.44 [1.44, 1.44]	1.44 [1.44, 1.45]	1.44 [1.44, 1.45]
$\sigma^{(A_{exs})}$ (%)	5.04 [5.03, 5.06]	4.95 [4.94, 4.96]	4.95 [4.94, 4.96]
SR	0.29 [0.29, 0.29]	0.29 [0.29, 0.29]	0.29 [0.29, 0.29]
$\mathbb{E}[\alpha]$ (%)	-0.0001 [-0.0001, -0.0001]	-0.0001 [-0.0001, -0.0001]	-0.0001 [-0.0001, -0.0001]

The box plot in Figure 4 indicates that the medians of the excess returns on either investing in the *DVA* provider, or in the portfolio having exactly the same investment policy as the *DVA* contract reference portfolio, are comparable. While excess returns on the financial market only are less volatile across scenarios, their maximum is lower than the best excess returns attainable via capital provision. Therefore, longevity risk exposure allows the equity holders to achieve higher excess returns in the best scenario, but entails greater downside risk due to the possible default of the *DVA* provider.

Figure 4: Box Plot of Equity Holders' Annualized Excess Return (%): $\theta = 20\%$
This figure presents the box plot of the equity holders' annualized return in excess of the risk-free rate (%), to either capital provision to the *DVA* provider (left), or investing in the reference portfolio (right). The reference portfolio is 20% invested in the risky stock index and 80% in the money market account. The line in the middle of the box is the median, while the edges of the box represent the 25^{th} and 75^{th} percentiles. The height of the box is the interquartile range, i.e., the interval between the 25^{th} and 75^{th} percentiles. The "+" symbols represent data points that are 1.5 times larger than the interquartile range.



6 Sensitivity Analysis

In this section, we carry out sensitivity analyses on the individual characteristics, stock exposure, the annuity provider's leverage, and the magnitude of longevity risk. These features influence the annuity provider's default rate and/or the volatility of the *GSA* benefits across scenarios and they subsequently alter the appeal of the *GSA* and the *DVA* to individuals.

6.1 Sensitivity to Risk Aversion

Individuals' preference for a *GSA* or a *DVA* is determined not only by the average level of benefits, but also by the risk on those benefits. Hence, individuals' preference depends on their risk aversion levels. We expect more risk-averse individuals to prefer the *DVA*, as long as the default rates entailed by their respective optimal *AIR*s are not too high. In this section, we consider highly risk-averse individuals with $\gamma = 10$, 15, and 20.

The optimal *AIR*s (Table 5) for highly risk-averse individuals and the annuity provider's default rates (Table 6, top panel) are comparable to those in the Baseline Case. Yet, in contrast to that case, the *CEL*s are positive (Table 6, middle panel). This suggests that individuals who are highly risk-averse prefer the *DVA* over the *GSA*, and are willing to pay a one-time loading of between 0.003% and 0.62% for the *DVA*. Despite that, when the annuity provider charges a loading equal to the *CEL*, equity holders attain Sharpe ratios that remain inferior to the 0.45 ratio of investing in the reference portfolio, and non-negative Jensen's alphas that are economically insignificant (Table 6, bottom panel).

Table 5: Highly Risk-Averse Individuals: Optimal *AIR*, h^* (%)

This table shows the optimal (AIR), Equation (5), of the *DVA* and *GSA* for individuals with risk aversion levels of $\gamma = 10$, 15, and 20. All other parameters are identical to those in the Baseline Case.

θ	γ		
(%)	**10**	**15**	**20**
0	3.56	3.58	3.59
20	4.44	4.25	4.04

Table 6: Highly Risk-Averse Individuals: Cumulative Default Rates (%), Certainty Equivalent Loading (CEL) (%) and Investment Performance Statistics
This top panel displays the Cumulative Default Rates, Equation (10), of the annuity provider. The middle panel shows the CEL, Equation (11), for individuals with risk aversion levels of $\gamma = 10$, 15, and 20. The bottom panel shows the Sharpe ratio and Jensen's alpha, Equation (12), when the loading is set at the CEL estimates in the middle panel. All other parameters are identical to those in the Baseline Case. The 99% confidence intervals are in parentheses.

Cumulative Default Rates (%)

θ	γ		
(%)	**10**	**15**	**20**
0	0.0106	0.0104	0.0104
20	0.0056	0.0064	0.0086

Certainty Equivalent Loading, CEL (%)

θ	γ		
(%)	**10**	**15**	**20**
0	0.037 [0.025, 0.049]	0.250 [0.233, 0.268]	0.410 [0.356, 0.458]
20	0.003 [-0.062, 0.069]	0.340 [0.095, 0.577]	0.620 [0.087, 1.145]

Sharpe Ratio and Jensen's alpha: Loading = CEL

θ	**Statistic**	γ		
(%)		**10**	**15**	**20**
0	*SR*	-0.0002 [-0.0039, 0.0034]	0.0083 [0.0046, 0.0119]	0.0146 [0.0110, 0.0183]
	$\mathbb{E}[\alpha]$ (%)	0 [-0.0000, -0.0000]	0.0003 [0.0003, 0.0003]	0.0005 [0.0005, 0.0005]
20	*SR*	0.292 [0.2920, 0.2920]	0.3062 [0.3062, 0.3062]	0.3171 [0.3171, 0.3171]
	$\mathbb{E}[\alpha]$ (%)	0 [0.0000, 0.0000]	0.0005 [0.0005, 0.0005]	0.0009 [0.0009, 0.0009]

6.2 Sensitivity to the Deferral Period

As the accuracy of longevity forecast depends on its horizon, the preference for either contract may depend on the age when the individual annuitizes. In the Baseline Case, individuals are aged 25 when purchasing a *DVA* contract or participating in the *GSA*. As retirement benefit payments commence at age 66, the deferral period is 40 years. Here, we shorten the deferral period by considering the situations where individuals decide between the *DVA* and the *GSA* at ages 45 and 65 instead (i.e., deferral periods of 20 years, and one year respectively).

When the deferral period is shorter, the *DVA* contract provider and *GSA* scheme administrator are able to make a more accurate forecast of survival probabilities. Thus, we expect smaller differences in the average level and standard deviation of benefits between contracts. However, this does not necessarily imply that the CEL estimates would be closer to zero, because the time-preference discounting, as governed by the subjective discount factor, β in Equation (2), plays a larger role when retirement is imminent. Therefore, while the difference between the benefits would be smaller, the effect in terms of utility would be greater.

Table 7 reveals that for individuals with risk-aversion levels of $\gamma = 5, 8$, the effect due to shorter time-discounting dominates the more accurate survival probability forecast; the CEL estimates are negative and more economically significant than those in the Baseline Case (Table 3). Thus, despite the smaller threat that longevity risk poses due to more accurate survival probability forecast, the imminence of retirement results in an increased preference for the *GSA* relative to the Baseline Case.

The least risk-averse individual, $\gamma = 2$, also has a stronger preference for the *GSA* than in the Baseline Case when the deferral period is 20 years, but this observation reverses when the deferral period is only one year. For the individual with $\gamma = 2$ who decides between the *DVA* and the *GSA* in the year prior to retirement, the higher accuracy of survival probabilities has a more prominent effect in utility terms than the shorter time-discounting has. This individual prefers the *GSA* to a lesser extent than in the Baseline Case because the CELs are less negative than that in the Baseline Case. Thus, apart from when the individual is less risk-averse and purchases an immediate annuity, the Baseline Case outcome stands.

Table 7: Deferral Period: Certainty Equivalent Loading (CEL) (%)
This top panel displays the *CEL*, Equation (11), for individuals aged 45 at annuitization, whereas the bottom panel corresponds to the *CEL*s for individuals aged 65 at that time. All other parameters are identical to those in the Baseline Case. The 99% confidence intervals estimated by the Delta Method are in parentheses.

20-year Deferral

θ	γ		
(%)	**2**	**5**	**8**
0	-0.380 [-0.386, -0.367]	-0.260 [-0.271, -0.252]	-0.150 [-0.161, -0.142]
20	-0.370 [-0.391, -0.350]	-0.270 [-0.293, -0.244]	-0.180 [-0.219, -0.140]

One-year Deferral

θ	γ		
(%)	**2**	**5**	**8**
0	-0.270 [-0.274, -0.266]	-0.230 [-0.234, -0.226]	-0.190 [-0.198, -0.190]
20	-0.260 [-0.262, -0.254]	-0.220 [-0.222, -0.213]	-0.190 [-0.192, -0.182]

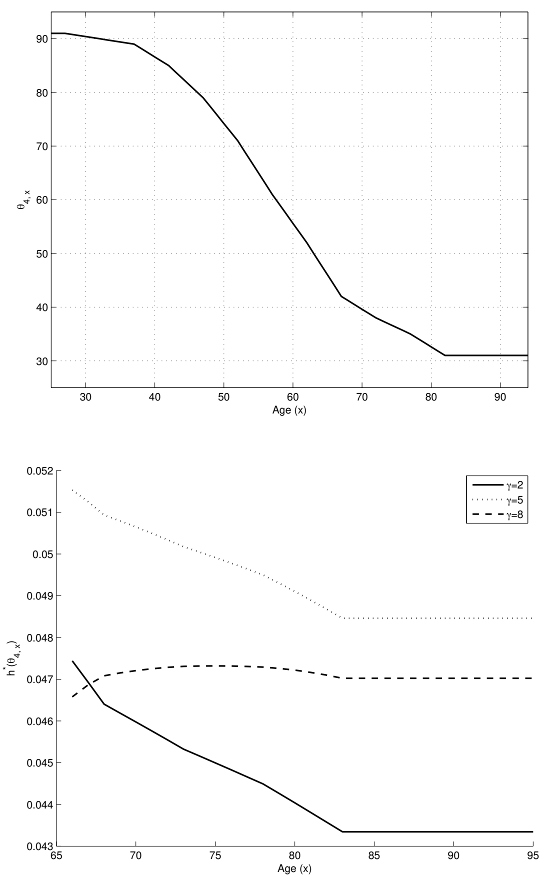
6.3 Sensitivity to the Stock Exposure

Both the *GSA* and the *DVA* contracts offer the *AIR* that maximizes individuals' expected utility given a fixed proportion invested in the stock index, θ, which we set to either 0% or 20%. We demonstrate that the implication of the Baseline Case, where individuals marginally prefer the *GSA*, holds as long as the allocation to the stock index corresponds to an *AIR* with similar default rates as those in the Baseline Case.

We consider four alternative exposures to the stock index. The first three are constant allocations over the planning horizon: $\theta_1 = 40\%$, $\theta_2 = 60\%$, $\theta_3 = \frac{\lambda_S}{\gamma \sigma_S}$. θ_3 corresponds to the individual's optimal exposure to stocks (Appendix A). For the least risk-averse individual ($\gamma = 2$), the optimal allocation to stocks, θ_3 is 147.2%. The moderately risk-averse individual ($\gamma = 5$) optimally invests 58.9% in the stock index whereas the most risk-averse individual ($\gamma = 8$) optimally invests 36.8% in stocks.

The fourth exposure that we consider, $\theta_4 = \{\theta_{4,x}\}_{x=25}^{95}$, is an age-dependent allocation that begins with around 90% allocation to stocks at age 25, which gradually diminishes to a minimum of about 30% post-retirement, until the maximum age (Figure 5, top panel).[22] A decreasing exposure to stocks as the individual grows older is consistent with popular financial advice (Viceira, 2001). In theory, when the investment opportunity set is constant, horizon-dependent investment strategies are optimal in situations where, for instance, the individual receives labor income (Viceira, 2001; Cocco et al., 2005), or where the individual's risk aversion parameter is time dependent (Steffensen, 2011). For all θs, the optimal *AIR* is set according to Equation (5), and summarized in Table 8. For the age-dependent θ_4, the optimal *AIR* also varies over the individual's life-span (Figure 5, bottom panel).

Figure 5: Glidepath Allocation to Stocks, θ_4 and the Optimal *AIR* (%)
The top panel shows the age-dependent allocation to stocks, defined on the industry average of Target-Date Funds in the U.S. in 2014 (Yang et al., 2016). The bottom panel displays the corresponding age-dependent optimal *AIR*, Equation (5), when the allocation to the stock index is θ_4.



For all θ_i, $i = 1, 2, 3, 4$, the optimal *AIR*s that correspond to these levels of

[22]This glidepath allocation is based on the 2014 Target-Date Fund industry average (Yang et al., 2016).

stock investments (Table 8) are slightly higher than those in the Baseline Case (Table 1), where $\theta = 20\%$. Due to the inverse relationship between the default rate and the *AIR*, the default rates are smaller than those in the Baseline Case. The *CEL* estimates in Table 8 imply that when investing according to θ_i, $i = 1, 2, 3, 4$, individuals marginally prefer the *GSA*, to a similar extent as they do in the Baseline Case. Therefore, the stock allocations that we consider preserve the individual preference for the *GSA* as in the Baseline Case.

Table 8: Exposure to Stock Index: Certainty Equivalent Loading (CEL) (%) The table displays the the optimal AIR $(h^*(\theta_i))$, Equation (5), the Cumulative Default Rates, Equation (10), and the CEL, Equation (11), when the underlying portfolio is θ_i invested in the stock index and $100-\theta_i$ invested in the money market account, for $i = 1, 2, 3, 4$. Estimates are calculated on 5 million replications. All other parameters are identical to those in the Baseline Case. The 99% confidence intervals estimated by the Delta Method are in parentheses.

$\theta_1 = 40\%$

Statistics	γ		
(%)	**2**	**5**	**8**
$h^*(\theta_1)$	4.59	5.06	4.72
Default Rates	0.0034	0.0020	0.0030
CEL	-0.340 [-0.344, -0.336]	-0.211 [-0.227, -0.195]	-0.080 [-0.218, 0.058]

$\theta_2 = 60\%$

Statistics	γ		
(%)	**2**	**5**	**8**
$h^*(\theta_2)$	5.08	5.24	4.26
Default Rates	0.0025	0.0021	0.0039
CEL	-0.344 [-0.348, -0.340]	-0.189 [-0.222, -0.155]	0.104 [-0.032, 0.240]

$\theta_3 = \frac{\lambda_S}{\gamma\sigma_S}$, optimal exposure

Statistics	γ		
(%)	**2**	**5**	**8**
θ_3	147.2	58.9	36.8
$h^*(\theta_3)$	0.0060	0.0052	0.0047
Default Rates	0.0003	0.0019	0.0026
CEL	-0.332 [-0.342, -0.321]	-0.186 [-0.235, -0.137]	-0.031 [-0.106, 0.043]

θ_4 = glidepath

Statistics	γ		
(%)	**2**	**5**	**8**
θ_4	Figure 5, top panel		
$h^*(\theta_4)$	Figure 5, bottom panel		
Default Rates	0.0052	0.0020	0.0022
CEL	-0.341 [-0.345, -0.336]	-0.307 [-0.390, -0.224]	-0.563 [-0.773, -0.352]

6.4 Sensitivity to the Level of Equity Capital

Individuals' preference for the *DVA* depends on the provider's default rates, determined by the level of the provider's capital buffer. We investigate the implication that the *DVA* provider's default rates has on individual preference by increasing the Baseline Case's leverage ratio by one standard deviation, comparable to halving the Baseline Case equity capital to 5%.[23]

When the equity capital is 5% of the value of liabilities instead of 10%, the default rates rise from 0.004-0.01% to 4.96-6.78% (Tables 2 and 9). The *CEL*s that accompany these high rates are economically significantly negative. For example, when $\theta = 20\%$, the most risk averse individual ($\gamma = 8$) is essentially indifferent between the *DVA* and *GSA* in the Baseline Case, but now values the *DVA* at only three-fourths of its best estimate price (Table 9, $CEL = -24\%$).

While the *CEL*s in the Baseline Case are similar for $\theta = 0$ and $\theta = 20\%$, they are noticeably more negative for $\theta = 20\%$ when the annuity provider has higher leverage. The amplified preference for the *GSA* in the presence of stock market risk is due to the resolution when a default occurs. When the annuity provider defaults, individuals recover the provider's residual wealth to purchase an equally weighted portfolio of bonds that mature in every remaining year of retirement until maximum age. This implies that individuals forgo mortality credit. In the case when $\theta = 20\%$, individuals additionally relinquish all potential reward from investing in the stock market. Relative to having the underlying portfolio fully invested in the money market account, when only mortality credit is lost, the consequence of default is more severe when the underlying portfolio is invested in the stock market. Therefore, in Table 9, the *CEL*s when $\theta = 20\%$ are considerably more negative than those when $\theta = 0$.

[23] The standard deviation of U.S. life insurers' leverage ratio between 1998-2011 is 3.7% whereas the average is around 90% (based on A.M. Best data from Koijen and Yogo, 2015). Using the definition of Leverage Ratio := 1 − Value of Equity/Value of Assets, the assumption that Value of Assets = Premium Collected + Value of Equity, and that the Premium Collected is normalized to 1, a 93.7% leverage ratio is equivalent to an initial capital of around 5%.

Table 9: Higher Leverage: Default Rates (%) and Certainty Equivalent Loading (CEL) (%)
This top panel displays the Cumulative Default Rates, Equation (10), of the annuity provider, whereas the bottom panel shows the CEL, Equation (11), when the level of equity capital is 5% of the present value of liabilities at the date of contract sale. All other parameters are identical to those in the Baseline Case. The 99% confidence intervals for the CEL are in parentheses.

Cumulative Default Rates (%)

θ	γ		
(%)	**2**	**5**	**8**
0	6.7826	6.4874	6.4092
20	5.6558	4.9730	4.9634

Certainty Equivalent Loading, CEL (%)

θ	γ		
(%)	**2**	**5**	**8**
0	-3.4 [-3.5, -3.4]	-5.5 [-5.6, -5.5]	-9.4 [-9.5, -9.3]
20	-5.6 [-5.7, -5.5]	-12.9 [-13.2, -12.7]	-24.0 [-24.3, -23.6]

When the annuity provider is more leveraged, the increased occurrence of defaults adversely affects the equity holders' excess return and its standard deviation, and makes exposure to longevity risk even less attractive for both $\theta = 0$ and $\theta = 20\%$. When $\theta = 0$, excess returns on the equity holders' investment is negative (Table 10, top panel). When the underlying portfolio is $\theta = 20\%$ invested in the stock market, higher leverage yields an excess return of 1.3%, lower than the 1.44% excess return of the Baseline Case (Tables 10 and 4, bottom panels). Due to the higher frequency of defaults, the standard deviation of excess return is around twice that of the Baseline Case (7.6% vs. 3.9% for $\theta = 0$; 9% vs. 5% for $\theta = 20\%$; Tables 4 and 10). As a result, the Sharpe ratio is halved whereas the Jensen's alphas are more negative than those in the Baseline Case.

Table 10: Higher Leverage: Equity Holders' Investment Performance Statistics
This table displays the equity holders' mean annualized return in excess of the risk-free rate of return ($\mathbb{E}\left[R^{(A_{exs})}\right]$, %), standard deviation of annualized excess return ($\sigma^{(A_{exs})}$, %), the Sharpe ratio (*SR*) and Jensen's alpha ($\mathbb{E}[\alpha]$, %), Equation (12), of capital provision to the *DVA* provider, when the level of equity capital is 5% of the present value of liabilities at the date of contract sale. The underlying portfolio is either invested in the money market account only ($\theta = 0$, top panel), or is 20% invested in the risky stock index, and 80% in the money market account ($\theta = 20\%$, bottom panel). The 99% confidence intervals are in parentheses.

$\theta = 0$

Statistic	γ		
	2	**5**	**8**
$R^{(A_{exs})}$ (%)	-0.085 [-0.089, -0.082]	-0.085 [-0.088, -0.082]	-0.085 [-0.088, -0.082]
$\sigma^{(A_{exs})}$ (%)	7.59 [7.57, 7.61]	7.56 [7.54, 7.58]	7.55 [7.53, 7.57]
SR	-0.011 [-0.015, -0.008]	-0.011 [-0.015, -0.008]	-0.011 [-0.0145, -0.008]
$\mathbb{E}[\alpha]$ (%)	-0.0009 [-0.0009, -0.0009]	-0.0009 [-0.0009, -0.0009]	-0.0009 [-0.0009, -0.0009]

$\theta = 20\%$

Statistic	γ		
	2	**5**	**8**
$R^{(A_{exs})}$ (%)	1.28 [1.28, 1.28]	1.29 [1.29, 1.30]	1.29 [1.29, 1.30]
$\sigma^{(A_{exs})}$ (%)	9.45 [9.42, 9.47]	9.22 [9.20, 9.25]	9.22 [9.19, 9.24]
SR	0.136 [0.132, 0.139]	0.1401 [0.137, 0.143]	0.140 [0.137, 0.143]
$\mathbb{E}[\alpha]$ (%)	-0.0009 [-0.0009, -0.0009]	-0.0009 [-0.0009, -0.0009]	-0.0009 [-0.0009, -0.0009]

6.5 Sensitivity to Longevity Risk

We investigate the effect of longevity risk on individual preference between the *GSA* and *DVA* via three other scenarios: we introduce parameter uncertainty surrounding the drift of the longevity time trend, and we adopt the Cairns et al. (2006) longevity model instead of the Lee and Carter (1992) model.

6.5.1 Drift Parameter Uncertainty of the Longevity Time Trend

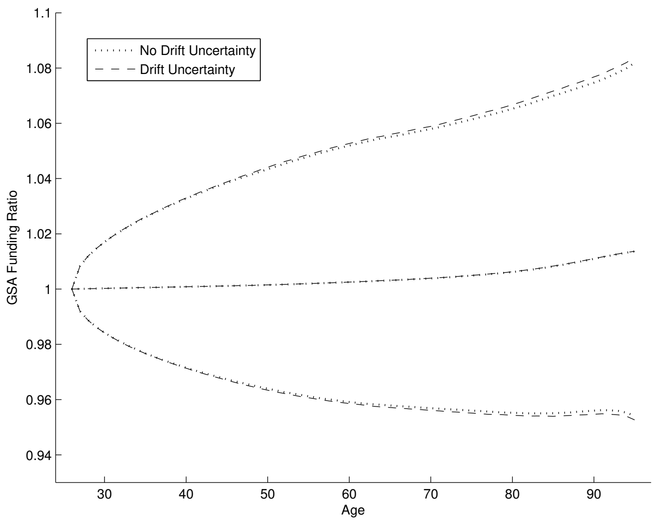
One way to depict the challenge of accurately estimating future survival probabilities is to introduce parameter uncertainty on the estimate of the drift term, $\hat{c}$, in the Lee and Carter (1992) model, Equation (8).

The maximum likelihood estimate for the drift term of the longevity model is Normally distributed, $\hat{c} \sim \mathscr{N}\left(c, \sigma_c^2\right)$. Based on the sample used for the model calibration, we obtain $\hat{c} = -1.0689$ and $\widehat{\sigma_c}$=0.0521. Without parameter uncertainty, the best m-year-ahead forecast at time t is $\widehat{k_{t+m}} = m\hat{c} + k_t$. To incorporate parameter uncertainty, we draw c_l from the distribution $\mathscr{N}\left(\hat{c}, \widehat{\sigma_c^2}\right)$ for the l^{th} simulation replication. The time trend governing longevity is thus $k_{t+m,l} = mc_l + k_{t,l} + \sum_{i=1}^{m} \varepsilon_{\delta,l}$, $\varepsilon_{\delta,l} \sim \mathscr{N}\left(0, \widehat{\sigma_\delta^2}\right)$, while the best m-year-ahead forecast relies on $\hat{c}$ as c_l is unobserved, i.e., $\widehat{k_{t+m,l}} = m\hat{c} + k_{t,l}$.

To intuitively gauge the implication of parameter uncertainty, we plot the mean, 5^{th} and 95^{th} percentiles of the *GSA* funding ratio prior to entitlement adjustments, with and without uncertainty over time (Figure 6). The *GSA* funding ratio reflects the entitlement adjustments. For instance, if the funding ratio is 1.02, then the *GSA* offers a benefit that is 2% higher than the entitlement in that year. Figure 6 suggests that parameter uncertainty has a faint effect on the benefits. The average entitlement adjustments are comparable to when the drift term is known with certainty. The only noticeable difference is that with parameter uncertainty, the 5^{th} and 95^{th} percentiles are slightly farther apart in the final years of retirement.

Figure 6: Drift Parameter Uncertainty: *GSA* Funding Ratio

This figure presents the mean, 5^{th} and 95^{th} percentiles of the funding ratio of a *GSA* prior to entitlement adjustments, for when longevity is modeled according to Lee and Carter (1992). When there is no parameter uncertainty surrounding the drift term of the longevity time trend, $c = \hat{c}$. When there is parameter uncertainty, $c \sim \mathcal{N}\left(\hat{c}, \widehat{\sigma_c^2}\right)$. The *GSA* is composed of individuals with a risk-aversion level of $\gamma = 5$ and the underlying portfolio is fully invested in the money market account. All other parameters are identical to those in the Baseline Case.



When there is uncertainty around the drift parameter, the *DVA* is disadvantaged by a higher default probability. However, the *GSA*'s appeal also diminishes as entitlement adjustments have a wider variation, relative to adjustments in the Baseline Case toward the end of retirement (Figure 6). Neither of these drawbacks is sufficiently decisive to sway individual preferences. Therefore, the *CEL*s deviate only marginally from those in the Baseline Case (Table 11).

Table 11: Drift Parameter Uncertainty: Cumulative Default Rates (%) and Certainty Equivalent Loading (CEL) (%)
The top panel presents the Cumulative Default Rates, Equation (10), of the annuity provider when there is parameter uncertainty surrounding the drift term of the longevity model's time trend. The bottom panel displays the CEL, Equation (11). All other parameters are identical to those in the Baseline Case. The 99% confidence intervals estimated by the Delta Method are in parentheses.

Cumulative Default Rates (%)

θ	γ		
(%)	**2**	**5**	**8**
0	0.0174	0.0144	0.0140
20	0.0066	0.0030	0.0030

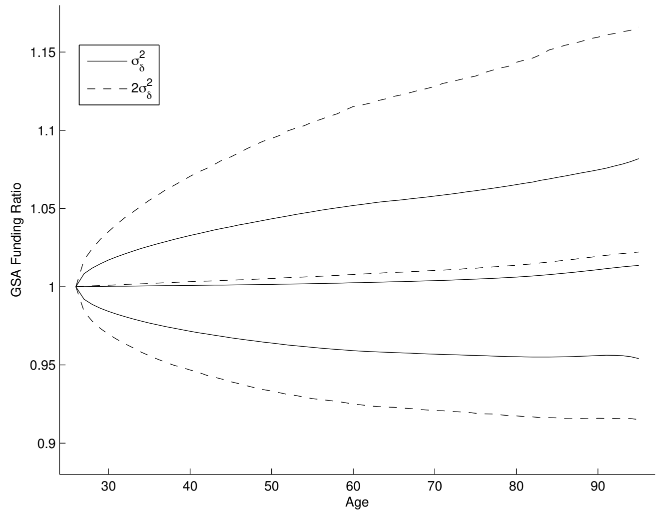
Certainty Equivalent Loading, CEL (%)

θ	γ		
(%)	**2**	**5**	**8**
0	-0.351 [-0.362, -0.339]	-0.193 [-0.205, -0.182]	-0.045 [-0.056, -0.033]
20	-0.345 [-0.356, -0.333]	-0.194 [-0.210, -0.178]	-0.046 [-0.089, -0.004]

6.5.2 Standard Deviation of the Longevity Time Trend Errors

We consider another prospect of longevity evolution with a variance of the longevity time trend that is twice the variance estimated from historical mortality data, i.e., σ_δ of Equation (8) is replaced by $\sqrt{2}\sigma_\delta = 3.562$. At a higher time trend standard deviation, the survival probabilities not only become more variable, but their conditional probabilities also decline (Denuit, 2009). Higher variability of survival outcomes is unfavorable both to the *GSA* participants, who bear larger variations in benefits, and to the *DVA* contract holders, due to the greater probability of default. On average, the entitlement adjustments under increased longevity risk are positive and higher than those in the Baseline Case, but there is also a wider variation in entitlement adjustments that rises in age (Figure 7).

Figure 7: *GSA* Funding Ratio
This figure presents the mean, 5^{th} and 95^{th} percentiles of the funding ratio of a *GSA* prior to entitlement adjustments, for when longevity is modeled according to Lee and Carter (1992). Parameters for the longevity model are either those in Figure 1 (denoted by σ_δ^2), or with the variance of the time trend error terms doubled (denoted by $2\sigma_\delta^2$). Individuals have a risk-aversion level of $\gamma = 5$ and the underlying portfolio is fully invested in the money market account. All other parameters are identical to those in the Baseline Case.



The default rates are considerably higher when the longevity time trend variance is doubled (i.e., 3.39-5.17%, Table 12). Consequently, individuals prefer the *GSA* to a large degree (Table 13, top panel).

Table 12: Doubled Longevity Time Trend Variance: Cumulative Default Rates (%)
This table displays the Cumulative Default Rates, Equation (10), of the annuity provider when the variance to the longevity model time trend is doubled. All other parameters are identical to those in the Baseline Case.

θ	γ		
(%)	**2**	**5**	**8**
0	5.1650	4.8704	4.8008
20	4.0596	3.4066	3.3928

Two factors govern individual preference. The first is the effect on the level and standard deviation of benefits; the second is the annuity provider's higher default risk due to less accurate longevity forecasts. To separately identify the two effects, we eliminate default risk by assuming a sufficiently high level of equity capital.

In the absence of default, the least risk-averse individual (i.e., $\gamma = 2$) marginally prefers the *GSA*, to a similar extent as she did in the Baseline Case (Table 13, middle panel). Thus, the least risk-averse individual's preference is invariant to the size of the standard deviation of the error terms, as long as the provider's default risk is unaffected. As for the more risk-averse individuals (i.e., $\gamma = 5$, 8), they prefer the *DVA* with no default risk and are willing to offer between 0.2% and 3.2% in loading for it. Thus, although a higher standard deviation to the longevity time trend errors transpires to a more volatile *GSA* benefit payment, making the *GSA* less appealing to individuals overall. Individuals who are at least moderately risk-averse would prefer the DVA only if the annuity provider's default risk were eliminated.

Despite the seemingly high loading that the annuity provider could charge on a *DVA* contract with no default risk, the loading is insufficient to yield equity holders a Sharpe ratio superior to the 0.45 ratio of investment without longevity risk exposure (Table 13, bottom panel). The Jensen's alpha is positive but economically insignificant. Therefore, longevity risk exposure does not improve the equity holders' risk-return tradeoff.

Table 13: Doubled Longevity Time Trend Variance: Certainty Equivalent Loading (CEL) (%) and Investment Performance Statistics

The top panel presents the CEL, Equation (11), when the variance of the errors of the longevity time trend is doubled, and the equity capital is 10% of the value of liabilities on the contract's date of sale. The middle panel displays the CEL in the same setting as in the top panel, but with the equity capital raised sufficiently to eliminate default risk. The bottom panel shows the Sharpe ratio (SR) and Jensen's alpha (α), Equation (12), when the loading is set at the CEL estimates in the middle panel. All other parameters are identical to those in the Baseline Case. The 99% confidence intervals are in parentheses.

CEL (%): With Default Risk

θ	γ		
(%)	**2**	**5**	**8**
0	-2.5 [-2.5, -2.5]	-3.2 [-3.2, -3.1]	-5.0 [-5.1, -4.9]
20	-3.9 [-3.9, -3.8]	-7.7 [-7.8, -7.5]	-15.9 [-16.4, -15.5]

CEL (%): No Default Risk

θ	γ		
(%)	**2**	**5**	**8**
0	-0.4 [-0.4, -0.4]	0.2 [0.1, 0.2]	0.7 [0.7, 0.7]
20	-0.3 [-0.4, -0.3]	3.2 [2.1, 4.2]	3.2 [3.1, 3.4]

Sharpe Ratio and Jensen's Alpha: No Default Risk, Loading = CEL

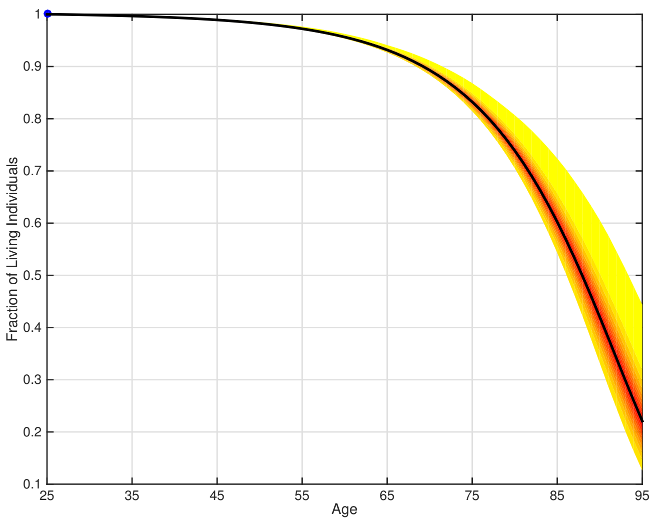
θ	**Statistic**	γ		
(%)		**2**	**5**	**8**
0	SR	0.0046 [0.0009, 0.0082]	0.0164 [0.0127, 0.0200]	0.0263 [0.0226, 0.0299]
	$\mathbb{E}[\alpha]$ (%)	0 [0.0000, 0.0000]	0.0001 [0.0001, 0.0001]	0.0002 [0.0002, 0.0002]
20	SR	0.4243 [0.4243, 0.4243]	0.4397 [0.4397, 0.4397]	0.4397 [0.4397, 0.4397]
	$\mathbb{E}[\alpha]$ (%)	0.0001 [0.0001, 0.0001]	0.0005 [0.0005, 0.0005]	0.0005 [0.0005, 0.0005]

6.5.3 Alternate Longevity Model

We next explore the choice of the longevity model by swapping the Lee and Carter (1992) model for the Cairns et al. (2006) model, which produces a wider range of survival probabilities at old age. We calibrate the Cairns et al. (2006) model over the same sample of mortality data as that described in Section 3.2. Figure 8 presents the fan plot of the simulated fraction of living individuals under the Cairns et al. (2006) model. The maximum range of the fraction of 25-year-olds still alive at older ages is 45% (i.e., at age 91), 50% more than the maximum range under the Lee and Carter (1992) model (i.e., 30% interval at age 88; Figure 2). This wider range translates into greater variability in benefits for the *GSA*, and higher default rates for the *DVA* provider.

Figure 8: Cairns et al. (2006) Fan Plot

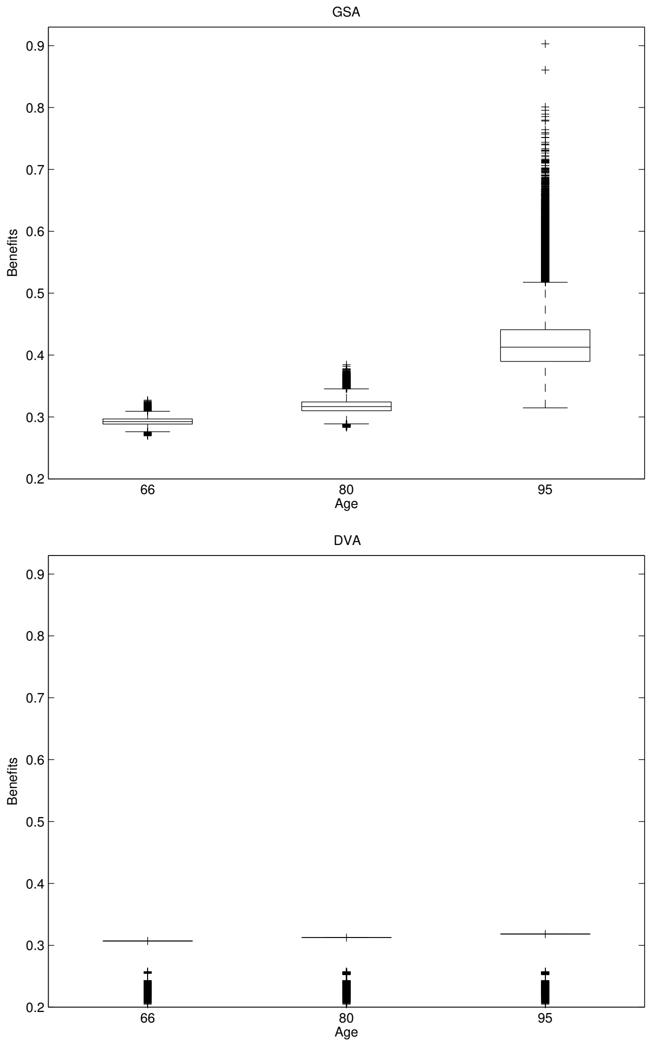
This figure presents the fan plot of the simulated fraction of living individuals (i.e., the population of 25-year-olds is normalized to one) over 10,000 replications when longevity is modeled according to the Cairns et al. (2006) model, which is calibrated on U.S. female death counts from 1980 to 2013 taken from the Human Mortality Database. Darker areas indicate higher probability mass.



With either the Lee and Carter (1992) or the Cairns et al. (2006) model, the rise in GSA benefits with age is accompanied by more uncertainty surrounding the benefits. However, the Cairns et al. (2006) model produces greater uncertainty as the individual ages, as seen by comparing the top panels in Figures 3 and 9. This generates greater individual preference for the *DVA* under the Cairns et al. (2006) model.

Figure 9: Box Plots of *GSA* and *DVA* Benefits: Cairns et al. (2006) Model

The figure presents the box plots of benefits for the *GSA* (top panel) and the *DVA* (bottom panel), accruing to an individual with a risk aversion level of $\gamma = 5$, at ages 66, 80 and 95. The underlying portfolio is invested in the money market account only. The line in the middle of the box is the median, while the edges of the box represent the 25^{th} and 75^{th} percentiles. The height of the box is the interquartile range, i.e., the interval between the 25^{th} and 75^{th} percentiles. The "+" symbols represent data points 1.5 times larger than the interquartile range.



For a fixed level of equity capital, the Cairns et al. (2006) model yields higher default rates because of the heightened uncertainty surrounding old age survival. If we maintain the Baseline Case's 90% leverage ratio, the default rates under the Cairns et al. (2006) model are between 0.48% to 2.21% (Table 14), substantially higher than the at-most 0.01% default rates when the Lee and Carter (1992) model is adopted (Table 2). Consequent to more defaults, individuals have a lower preference for the *DVA* (Table 14, bottom panel), as the *CEL* estimates are more negative than those in the Baseline Case (Table 3). Therefore, individuals prefer the *DVA* contract under the Cairns et al. (2006) model only if the associated default risk is curtailed. Regardless of whether equity holders provide enough capital to eliminate default risk, the Sharpe ratio of equity provision is lower than the ratio of abstaining from longevity risk exposure. The Jensen's alpha of equity provision is positive but economically insignificant.

Table 14: Cairns et al. (2006) Mortality Model with Default: Cumulative Default Rates (%) and *CEL* (%)
The top panel presents the Cumulative Default Rates, Equation (10), whereas the bottom panel displays the *CEL*, Equation (11), when life expectancy follows the Cairns et al. (2006) model, calibrated to the same sample as the Lee and Carter (1992) model. All other parameters are identical to those in the Baseline Case. The 99% confidence intervals estimated by the Delta Method are in parentheses.

Cumulative Default Rates (%)

θ	γ		
(%)	**2**	**5**	**8**
0	2.2120	1.8082	1.7120
20	0.9676	0.4808	0.4756

CEL (%)

θ	γ		
(%)	**2**	**5**	**8**
0	-0.950 [-0.970, -0.930]	-0.660 [-0.690, -0.630]	-0.975 [-1.025, -0.924]
20	-0.877 [-0.906, -0.847]	-0.503 [-0.571, -0.436]	-1.515 [-1.763, -1.268]

Additionally, the choice of the longevity model underlies the inference of Maurer et al. (2013). While we find that individuals marginally prefer the *GSA*, Maurer et al. (2013) observe the opposite – the *CEL* for the contract indexed to longevity is positive (Table 7 of Maurer et al., 2013). When we assume that no default occurs, as do Maurer et al. (2013), we are able to reconcile our results to theirs (i.e., individuals who are moderately risk-averse to risk-averse, $\gamma = 5$ and 8, prefer the *DVA*; Table 15, top panel). The most risk-averse individual is willing to pay as much as 1% in loading to shed longevity risk. Despite that, when the annuity provider sets the loading to be equal to the *CEL*, the accompanying Sharpe ratio remains inferior to the Sharpe ratio of investing in only the financial market, i.e., 0.45 when $\theta = 20\%$, whereas the Jensen's alpha is positive but economically insignificant

(Table 15, bottom panel). Therefore, while individual preference is sensitive to the choice of the longevity model, the extent that individuals are willing to pay to insure against longevity risk is insufficient to entice equity holders to be exposed to longevity risk.

Table 15: Cairns et al. (2006) Mortality Model with No Default: Certainty Equivalent Loading (CEL) (%) and Investment Performance Statistics
The top panel presents the CEL, Equation (11), when life expectancy follows the Cairns et al. (2006) model is calibrated to the same sample as the Lee and Carter (1992) model. The bottom panel shows the Sharpe ratio (SR) and Jensen's alpha (α), Equation (12), when the loading is set at the CEL estimates in the top panel. Equity capital is sufficiently high such that no default occurs. All other parameters are identical to those in the Baseline Case. The 99% confidence intervals are in parentheses.

CEL (%)

θ (%)	γ		
	2	**5**	**8**
0	-0.089 [-0.099, -0.079]	0.528 [0.519, 0.537]	1.019 [1.011, 1.028]
20	-0.092 [-0.101, -0.082]	0.461 [0.448, 0.475]	0.874 [0.835, 0.913]

Sharpe Ratio and Jensen's Alpha: No Default Risk, Loading = CEL

θ (%)	**Statistic**	γ		
		2	**5**	**8**
0	SR	0.0206 [0.0170, 0.0242]	0.0481 [0.0444, 0.0517]	0.0701 [0.0665, 0.0738]
	$\mathbb{E}[\alpha]$ (%)	0.0001 [0.0001, 0.0001]	0.0002 [0.0002, 0.0002]	0.0002 [0.0002, 0.0002]
20	SR	0.4337 [0.4337, 0.4337]	0.4362 [0.4362, 0.4362]	0.4379 [0.4379, 0.4379]
	$\mathbb{E}[\alpha]$ (%)	0.0001 [0.0001, 0.0001]	0.0001 [0.0001, 0.0001]	0.0002 [0.0002, 0.0002]

7 Conclusion

We investigate longevity risk management in retirement planning in the presence of two alternatives: individuals participating in a collective scheme that adjusts retirement income according to longevity evolution, or purchasing a variable annuity contract offered by an equity-holder-backed annuity provider. Our model features the perspective of not only the individuals, who evaluate their welfare in retirement, but also the equity holders, who weigh their risk-return tradeoff from longevity risk exposure.

Due to the entitlement adjustments arising from errors in survival probability forecasts, the collective scheme provides more volatile benefits than those of an annuity contract. However, the collective scheme also offers a slightly higher level of benefits on average because, for errors of the same magnitude, over- and under-estimating the log central death rates has asymmetric effects on the benefits. The annuity contract provider relies on limited equity capital to subsume forecasting errors, and so is subject to default risk. Although the annuity contract shields individuals from downward entitlement adjustments up to a limit, it deprives individuals of any upward adjustments, as these gains belong to the equity holders.

We find that individuals marginally prefer the collective scheme over the annuity contract priced at its best estimate. This implies that the annuity provider is unable to charge a positive loading on the contract, subsequently failing to compensate its equity holders for bearing longevity risk. Therefore, when individuals have the choice to form a collective scheme, the annuity provider without any advantage bearing longevity risk and has to fully hedge financial market risk would not exist in equilibrium. Our finding is robust to numerous individual characteristics, stock market risk exposure, and heightened uncertainty surrounding life expectancy.

Our results advocate for collective mechanisms in pension provision. The pressing issue of population aging, and the gradual maturation of the longevity risk derivatives market, may spur reform, e.g., the U.S. Chamber of Commerce (2016) recommends new plan designs to enhance the private retirement system. Collective schemes may serve as a benchmark that the annuity contract has to match or surpass with respect to the individuals' expected utility.

One limitation of our work is the exclusion of channels that may reduce the insurer's effective longevity exposure, such as synergies in product offering (e.g., natural hedging of longevity risk via the sale of annuities and life insurance contracts; Wong et al., 2017), access to reinsurance (Baione et al., 2016) and shadow insurance (Koijen and Yogo, 2016). There are also alternative resolution mechanisms in the case of default, and other factors that may influence annuitization decisions, such as bequest motives, medical expenses, social security, uninsurable income, etc. (Lockwood, 2012; Pashchenko, 2013; Peijnenburg et al., 2017; Ai et al., 2016; Yogo, 2016). Examining these features in future research could generate more insights into the management of longevity risk in retirement planning.

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Appendices

A Rationale of the Contract Definition

The *DVA* and *GSA* contracts are not only modeled along the variable annuity contracts studied in the literature (Koijen et al., 2011; Maurer et al., 2013), but are also relatable to an individual's optimal consumption and investment.

The problem of optimal consumption and investment is composed of two separate parts: the allocation of initial wealth over each retirement year, and the investment strategy. Aase (2015) shows that for an expected-CRRA-utility-maximizing individual facing mortality and stock market risks, the optimal allocation of initial wealth decays geometrically in the retirement horizon. The *AIR* in our setting represents precisely this rate of decay.

When individuals are subject to longevity risk, its existence would not change the optimal wealth and asset allocation; what would complicate the solution is the ability to react to longevity evolution (Huang et al., 2012). We, however, assume that the contract's parameters are deterministic (i.e., fixed in the year when it is sold, and the incorporation of new information thereafter is prohibited). Therefore, by an appropriate choice of the *AIR*, h^*, the contract described by Equations (3) and (4) coincides with the optimal decumulation path of the individual.

We next solve the utility maximization problem, (13), to obtain the optimal *AIR* and investment strategy for a contract defined by Equations (3) and (4).

At time t_0, the individual purchases the maximum number of variable annuity contracts affordable with a lump sum capital normalized to one. The annuity contract commences benefit payment in year t_R, until the year T, conditional on the individual's survival. In the financial market setting as described in Section 2.1, with a deterministic fraction of wealth $\theta = \{\theta_t\}_{t=t_0}^{T}$ invested in the risky stock index, and $1-\theta$ invested in the money market account, the value of the reference portfolio evolves according to $\frac{dW_t}{W_t} = (r + \theta_t \lambda \sigma_S)\,\mathrm{d}t + \theta_t \sigma_S\,\mathrm{d}Z_{S,t}$.

$$\max_{\{\theta_t, h(t,\theta_t)\}_{t=t_R}^{T}} \mathbb{E}_{t_0}\left[U(\Xi)\right] \quad (13)$$

$$= \mathbb{E}_{t_0}\left[\int_{t_R}^{T} \mathrm{e}^{-\beta t} \frac{\Xi_t^{1-\gamma}}{1-\gamma}\left(\Pi_{s=t_0}^{t}\, {}_1p_{x+(s-t_0)}^{(s)}\right) \mathrm{d}t\right]$$

$$\Xi_t = \begin{cases} \frac{1}{A(h)}\,\mathrm{e}^{-h(t,\theta_t)\times(t-t_R)}\,\frac{W_t}{W_{t_0}} & \text{if alive in year } t \\ 0 & \text{otherwise} \end{cases} \quad (14)$$

$$A(h) = \int_{t_R}^{T} \mathrm{e}^{-h(t,\theta_t)\times(t-t_R)}\,\mathbb{E}_{t_0}\left[\left(\Pi_{s=t_0}^{t}\, {}_1p_{x+(s-t_0)}^{(s)}\right)\right] \mathrm{d}t \quad (15)$$

$h(t,\theta_t)$ = Assumed Interest Rate

β = subjective discount factor

γ = risk aversion parameter

W_t = value of the reference portfolio with the investment policy θ

$$\mathbb{E}_{t_0}\left[\Pi_{s=t_0}^{t}\, {}_1p_{x+(s-t_0)}^{(s)}\right] = {}_{t-t_0}p_x^{(t_0)}$$

$A(h)$ is the cost per unit of a zero-loading contract. It is straightforward to verify that the contract has a present expected value of one for any $h \in \mathbb{R}^{T-t_R}$, and thus satisfies the budget constraint. Given any θ, the first order condition, $\mathbb{E}_{t_0}\left[U(\Xi)\right]/\partial h = 0$ yields the optimal *AIR*, Equation (16).

$$h^*(t,\theta_t) = r + \frac{\beta - r}{\gamma} - \frac{1-\gamma}{\gamma}\theta_t \sigma_S \left(\lambda_S - \frac{\gamma \theta_t \sigma_S}{2}\right) \quad (16)$$

r = constant short rate

β = subjective discount factor

γ = risk aversion parameter

θ_t = fraction of wealth allocated to the stock index at time t, $t_R \leq t \leq T$

σ_S = standard deviation governing the stock index's dynamics

λ_S = instantaneous Sharpe ratio of the stock index

Equation (16) is composed of the risk-free rate, the difference between the subjective discount factor and the risk-free rate, adjusted by the risk aversion parameter, and a term concerning the exposure to the stock index, weighted by the risk aversion level.

If the returns on the investment were constant at r (e.g., either $\theta = 0$ or $\sigma_S = 0$), for any given level of the elasticity of inter-temporal substitution, γ, the shape of the optimal consumption path depends on the relative magnitude of β and r. An individual who discounts future consumption at a higher rate than the constant interest rate (i.e., $\beta > r$, an impatient individual) prefers a downward sloping consumption path whereas a more patient person (i.e., $\beta < r$) optimally chooses an upward sloping path. When $\theta \neq 0$ and $\sigma_S \neq 0$, then the risk aversion level, the standard deviation and the market price of stocks also have a role in determining the optimal consumption path.

The first-order condition corresponding to the allocation to the stock index, $\mathbb{E}_{t_0}[U(\Xi)]/\partial\theta = 0$, implies the optimal allocation to the risky asset:

$$\theta^* = \frac{\lambda_S}{\gamma\sigma_S} \quad (17)$$

The optimal allocation to the stock index, θ^*, is independent of time and wealth, and is identical to the optimal investment policy of Merton (1969).

The variable annuity contract provides the optimal decumulation path when the *AIR* is set to $h^*(t, \theta_t^*)$. By prohibiting the incorporation of new information into the contract definition after its date of sale (i.e., enforcing deterministic, but possibly time-varying contract parameters), longevity risk does not influence the optimal *AIR* and the optimal portfolio choice.

The conception of the *GSA* as a collective justifies the assumption that it prioritizes individual welfare (i.e., maximizes individuals' expected utility in retirement). Therefore, the *GSA* offers an *AIR* that is in the best interest of the individuals, without conflict among its stakeholders. As for the annuity provider, such contracts are also conceivable. For instance, Froot (2007) suggests that insurers should shed all liquid risks for which they have no comparative advantage to outperform (e.g., financial market risk), and devote their entire risk budget to insurance risks (e.g., longevity risk). The selling of variable annuities without any financial guarantee achieves precisely this goal. Besides, Gatzert et al. (2012) demonstrate that if an insurance company sets contract parameters for a participating life insurance contract such that they maximize the contract's value (e.g., expected utility) to the individual, the individual may be more willing to pay more for the contract. Therefore, the provision of contracts defined according to Equations (3) and (4) under either a cooperative setup or by a for-profit entity is plausible.

B Definition of the Book Value of Liabilities

Suppose that the *DVA* provider or the *GSA* administrator issues contract(s) to a cohort who is aged x at time t_0, promising entitlements of $\Xi^K(h^*, F, t, x)$, $K \in \{DVA, GSA\}$, in every year t, $t_R \leq t \leq T$, conditional on the individual's survival. The estimate of the entity's book value of liabilities at time t, $t_0 \leq t \leq T$, is:

$$L_t := \Xi^K(h^*, F, t, x) \int_{s=\max\{t_R, t\}}^{T} \exp\left(-h^*(s, \theta) \times (s-t)\right) \times \ldots$$
$$_{s-t}p_{x+t-t_0}^{(t)} \mathrm{ds} \tag{18}$$

$$\begin{aligned}
_{s-t}p_{x+t-t_0}^{(t)} &= \text{conditional probability in year } t \text{ that a living individual of age } x+t \\
&\quad \text{lives for at least } s-t \text{ more years} \\
h^*(t, \theta) &= \text{Optimal } AIR\text{, Equation (5)} \\
\Xi^K(h^*, F, t, x) &= \text{benefit at time } t \text{ for contract } K \in \{GSA, DVA\} \\
&= \begin{cases} \text{Equation (6)} & \text{if } K = DVA \\ \text{Equation (7)} & \text{if } K = GSA \end{cases}
\end{aligned}$$

B.1 Illustration of the Case with No Risk

To motivate the definition of Equation (18), let us consider a three-period case ($t = t_0, t_1, t_2$) in the absence of stock market and longevity risks. Assume that the individual buys exactly one unit of the retirement contract at retirement in year t_0, lives with certainty to collect the benefits in year $t_1 = t_0 + 1$, and dies with certainty before the year $t_2 = t_1 + 1$. Suppose that the reference portfolio is fully invested in the money market account, earning an interest rate that is constant at 2%. Furthermore, we adopt a constant AIR, $h = 3\%$, and zero contract loading, $F = 0$. As there is no uncertainty in this example, Equation (18) should yield precisely the value of liabilities at time t.

By definition of the DVA contract, there are two payments to be made: one in the year t_0 and another in the year t_1. The payment in t_0 is:

$$\begin{aligned}
\Xi(h, 0, t_0, x) &= 1 \times \frac{W_{t_0}^{Ref}}{W_{t_0}^{Ref}} e^{-h\times(t_0-t_0)} \\
&= 1
\end{aligned}$$

The second payment, in present value at time t_1 is:

$$\begin{aligned}
\Xi(h, 0, t_1, x) &= 1 \times \frac{W_{t_1}^{Ref}}{W_{t_0}^{Ref}} e^{-h\times(t_1-t_0)} \\
&= \frac{W_{t_0}^{Ref} \times e^{0.02}}{W_{t_0}^{Ref}} e^{-h} \\
&= e^{-h+0.02} \\
&= e^{-0.01}
\end{aligned} \tag{19}$$

Discounting Equation (19) by the constant interest rate, we obtain the present value at time t_0, of the payment due at time t_1:

$$\begin{aligned} PV_{t_0}\left[\Xi(h,0,t_1,x)\right] &= \Xi(h,0,t_1,x)\times e^{-0.02\times(t_1-t_0)} \\ &= e^{-0.01-0.02} \\ &= e^{-0.03} \end{aligned}$$

The present value of liabilities at time t_0 is

$$\Xi\left(h,0,t_0,x\right)+PV_{t_0}\left[\Xi(h,0,t_1,x)\right] = 1+e^{-0.03} \tag{20}$$

It remains to show that Equation (18) yields Equation (20):

$$\begin{aligned} L_t &= \Xi(h,0,t_1,x)\times\left(e^{-h\times 0}\,{}_0p_t^{(t)}+e^{-h\times 1}\,{}_1p_t^{(t)}\right) \\ &= 1\times\left(1+e^{-h}\right) \\ &= 1+e^{-0.03} \end{aligned}$$

B.2 Illustration of the General Case

We price the liabilities of the pension provision entity by constructing a replicating portfolio for its contractual obligation. We demonstrate that the price of the portfolio that replicates all the cash flows of an annuity contract is Equation (18).

In the setting with longevity but no mortality risk, we consider the liability associated with a contract holder who purchased $\frac{1}{A}$ unit(s) of contracts when aged x in the year $t_0=0$, retired in the year $t=t_R$, while being subject to unknown survival probabilities throughout the horizon, until the maximum age in the year $t=T$, when death is certain.

The pension provision entity is contractually obliged to make annual benefit payments from the individual's retirement in the year $t=t_R$ until he or she attains maximum age in the year $t=T$, conditional on her survival. Let W_t^{Ref} be the price at time t of the reference portfolio to which the benefits are indexed, $t\in[t_0,T]$.

Absent longevity risk, by purchasing the sum of all the units of the reference portfolio in Column (2) of Table 16 at time t, the annuity provider would be able to fulfill its contractual obligation with certainty. For instance, to meet the payment at time t_R, the annuity provider purchases $\frac{1}{A}\frac{1}{W_{t_0}^{Ref}}e^{-h\times 0}\,{}_{t_R-t}p_x^{(t_0)}$ units of the reference portfolio at time t_0. When longevity risk is absent, the conditional expectation, made at time t_0, of the individual's survival in year t_R coincides with the realized survival probability, i.e., ${}_{t_R-t}p_x^{(t_0)}={}_{t_R-t}p_x$. The value of this portfolio will evolve

Table 16: Future Cash Flow and the Best Replicating Portfolio of the Pension Provision Entity
This table presents the entitlements due in each year of retirement until maximum age, in future value of the year when the entitlements are due (column (1)), and the corresponding Best Replicating Portfolio in terms of units of the reference portfolio (column (2)). The Best Replicating Portfolio is obtained by taking the conditional expectation of the benefits in future value.

Time	**Benefits in Future Value**	**Best Replicating Portfolio (constructed at time t)** Units of the Reference Portfolio to purchase at time t
	(1)	(2)
t_R	$\frac{1}{A}\frac{W^{Ref}_{t_R}}{W^{Ref}_{t_0}}\,\mathrm{e}^{-h(t_R-t_R)} \times \Pi_{l=t_0}^{t_R-1}\ {}_1p_{x+l-t_0}$	$\frac{1}{A}\frac{1}{W^{Ref}_{t_0}}\,\mathrm{e}^{-h(t_R-t_R)}\ {}_{t_R-t}p^{(t)}_{x+t-t_0}$
t_R+1	$\frac{1}{A}\frac{W^{Ref}_{t_R+1}}{W^{Ref}_{t_0}}\,\mathrm{e}^{-h(t_R+1-t_R)} \times \Pi_{l=t_0}^{t_R}\ {}_1p_{x+l-t_0}$	$\frac{1}{A}\frac{1}{W^{Ref}_{t_0}}\,\mathrm{e}^{-h(t_R+1-t_R)}\ {}_{t_R+1-t}p^{(t)}_{x+t-t_0}$
t_R+2	$\frac{1}{A}\frac{W^{Ref}_{t_R+2}}{W^{Ref}_{t_0}}\,\mathrm{e}^{-h(t_R+2-t_R)} \times \Pi_{l=t_0}^{t_R+1}\ {}_1p_{x+l-t_0}$	$\frac{1}{A}\frac{1}{W^{Ref}_{t_0}}\,\mathrm{e}^{-h(t_R+2-t_R)}\ {}_{t_R+2-t}p^{(t)}_{x+t-t_0}$
$\vdots$	$\vdots$	$\vdots$
T	$\frac{1}{A}\frac{W^{Ref}_{T}}{W^{Ref}_{t_0}}\,\mathrm{e}^{-h\times(T-t_R)} \times \Pi_{l=t_0}^{T-1}\ {}_1p_{x+l-t_0}$	$\frac{1}{A}\frac{1}{W^{Ref}_{t_0}}\,\mathrm{e}^{-h(T-t_R)}\ {}_{T-t}p^{(t)}_{x+t-t_0}$

along with the financial market, to be worth exactly $\frac{1}{A}\frac{W_{t_R}^{Ref}}{W_{t_0}^{Ref}} \times \Pi_{l=t_0}^{t_R-1}\, {}_{l-t_0}p_{x+l-t_0}^{(l)}$, the payment due at time t_R. By the same reasoning for the rest of the entries in Column (2), Equation (21) is thus the total units of the reference portfolio to be held at any time t, such that the pension provision entity fully hedges financial market risk.

$$\int_{s=\max\{t_R,t\}}^{T} \frac{1}{A}\frac{1}{W_{t_0}^{Ref}} e^{-h(s-t_R)}\, {}_{s-t}p_{x+t-t_0}^{(t)} \mathrm{ds} \tag{21}$$

Equation (21) is an estimate of the liabilities at time t, in terms of the *units* of reference portfolio. Each unit is worth W_t^{Ref} at time t. To obtain the *value* of liabilities, we take the portfolio's corresponding value:

$$W_t^{Ref} \quad \times \quad \int_{s=\max\{t_R,t\}}^{T} \frac{1}{A}\frac{1}{W_{t_0}^{Ref}} e^{-h(s-t_R)}\, {}_{s-t}p_{x+t-t_0}^{(t)} \mathrm{ds} \tag{22}$$

As $\Xi(h,F,t,x) = \frac{1}{A}\frac{W_t^{Ref}}{W_{t_0}^{Ref}} e^{-h(t-t_R)}$ by definition, we can substitute it into Equation (22) to get

$$\begin{aligned} L_t \quad &:= \quad \Xi(h^*,F,t,x)\int_{s=\max\{t_R,t\}}^{T} \exp\left(-h^*(s,\theta)\times(s-t)\right)\times\ldots \\ &\qquad {}_{s-t}p_{x+t-t_0}^{(t)} \mathrm{ds} \end{aligned} \tag{23}$$

Equation (23) is identical to Equation (18).

When there is longevity risk, the Best Replicating Portfolio is identical to column (2) of Table 16, but this best estimate may not necessarily provide the exact cash flow to meet the annuity provider's contractual obligations because the realized survival probability may deviate from its conditional expectation made at time t, which then triggers the provider's default.

C Delta Method

We apply the Delta Method (Theorem 5.5.4 of Casella and Berger, 2002) to estimate the variance of the CELs, which is used to compute their confidence intervals.

Consider the function $g(x,y) = \left(\frac{x}{y}\right)^{\frac{1}{\gamma-1}} - 1$. By the definition of Equation (11), $CEL = g\left(U\left(\Xi^{GSA}\right), U\left(\Xi^{DVA}\right)\right)$. We estimate the CEL by plugging the expected utility into $g(.)$, $g\left(\mathbb{E}_0\left[U\left(\Xi^{GSA}\right)\right], \mathbb{E}_0\left[U\left(\Xi^{DVA}\right)\right]\right)$. Theorem 5.5.24 of Casella and Berger (2002) suggests the following estimate for its variance:

$$\begin{aligned}
\text{Var}\left\{g\left(\mathbb{E}_0\left[U\left(\Xi^{GSA}\right)\right],\ldots\right.\right. &= g_x^2\text{Var}\left(U\left(\Xi^{GSA}\right)\right)+g_y^2\text{Var}\left(U\left(\Xi^{DVA}\right)\right)+\ldots \\
\left.\left.\mathbb{E}_0\left[U\left(\Xi^{DVA}\right)\right]\right)\right\} & \quad 2g_xg_y\text{cov}\left(U\left(\Xi^{GSA}\right),U\left(\Xi^{DVA}\right)\right) \qquad (24)\\
g_x &= g_x\left(\mathbb{E}_0\left[U\left(\Xi^{GSA}\right)\right],\mathbb{E}_0\left[U\left(\Xi^{DVA}\right)\right]\right) \\
g_y &= g_y\left(\mathbb{E}_0\left[U\left(\Xi^{GSA}\right)\right],\mathbb{E}_0\left[U\left(\Xi^{DVA}\right)\right]\right)
\end{aligned}$$

g_x and g_y denote the first partial derivative of $g(.)$ with respect to x and to y respectively. $\text{Var}\left(U\left(\Xi^K\right)\right)$ for $K \in \{GSA, DVA\}$ and $\text{cov}\left(U\left(\Xi^{GSA}\right),U\left(\Xi^{DVA}\right)\right)$ are estimated by the sample variance and sample covariance.