A value-based approach to the redesign of US state pension plans[*]

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Abstract

We explore the financial sustainability of a typical U.S. state civil servants pension fund under the continuation of current policies and under alternative policies, such as changes in contribution, indexation and investment allocation policies. Applying the value-based asset-liability management method, we find that all participant cohorts derive substantial net benefit from the current pension contract, while all tax-paying cohorts have to make substantial contributions. The proposed policy adjustments can alleviate a substantial part of the burden on tax payers, although at the cost of the fund participants. Especially a policy of conditional inflation indexation seems promising. It is worth noting that our use of value-based asset-liability management can be applied to many more examples of pension reform.

Keywords: U.S. state civil servants pension funds, underfunding, value-based asset-liability management, contributions, benefits, indexation, participants, tax payers.

JEL Codes: G23, H60.

1 Introduction

In the past decades the U.S. has witnessed a trend away from defined-benefit (DB) towards defined-contribution (DC) pension plans. However, an exception to this trend are the state plans that manage the pensions of state civil servants. Despite the aging of the population, these plans still largely operate on a DB basis, although it is clear that in many cases the financial situation of the fund is too weak to fully honor the pension promises made to its participants. However, these promises cannot so easily be reneged upon (see Brown and Wilcox (2009)). In fact, they may even get priority to the states' debt holders when a state goes bankrupt. Hence, in many instances existing pension promises to state civil servants threaten the financial health of the state, possibly resulting in large claims on its tax payers and/or a crowding out of public services.

This paper explores the financial sustainability and redistributional features of a typical U.S. state defined-benefit (DB) pension plan under unchanged fund policies and under alternative policies in terms of contributions, benefits or indexation aimed at alleviating the financial burden on the tax payer. We also explore different changes in the fund's investment mix. We assume that pension promises cannot be defaulted upon. We apply the so-called method of value-based asset-liability management (value-based ALM) to address reform-induced value redistribution among the different stakeholders of the fund, i.e. the various cohorts of plan participants and tax payers. Essentially, the method involves rewriting a pension plan in terms of embedded options held by the fund's stakeholders. By providing a market-based valuation of all cash flows associated with the pension contract, the method demonstrates the consequences for the different stakeholders of the various interventions. Policy changes are always a zero-sum game, implying that the total value of the contract to all stakeholders together is unchanged and that policy changes can only shift value among groups of stakeholders. To the best of our knowledge, our paper is the first attempt at applying value-based ALM to the study of reform-induced redistribution within U.S. state pension funds.

We simulate a representative pension fund over a period of three-quarters of a century. This is quite a common horizon for official program projections. Given the substantial economic and institutional uncertainties over the very long run, it does not seem meaningful to extend the horizon further. The standard or "classic" ALM results reveal that raising the share of the amortization cost that is contributed, speeding up the amortization payment or halving indexation to consumer-price inflation leads to a substantial long-term improvement of the funding ratio, i.e. the value of the fund's assets over its liabilities. A policy that is of particular interest is a policy of inflation indexation that is conditional on the funding ratio. A funding ratio below one leads to less-than-full or even zero indexation, while a funding ratio above one leads to full indexation or even more. Such type of indexation policy has become popular among Dutch pension funds. This way of using indexation has a tendency of improving the financial position of the fund, but at the same time compressing the spread in the distribution of possible

funding ratios and, thereby, also compressing the spread in the distribution of the contribution rate, as its amortization component is linked to the funding ratio. Hence, a well-designed policy of conditional indexation may be of particular interest for policymakers concerned with a redesign of the U.S. system of state pension plans. It facilitates pension-fund supervision, as intervention is less frequently needed. For the same reason, it also facilitates the governance of the fund. Moreover, its continued existence is more secure by avoiding low funding ratios that give especially the young participants an incentive to leave the fund if they have a chance to (see Beetsma et al. (2012)).

Our value-based ALM results show that the current pension contract yields a substantial net benefit to all cohorts of fund participants, which in turn implies a substantial financial burden on all cohorts of tax payers. In present value terms, over the 75-years horizon under the base plan the fund participants receive almost 7 trillion dollars in net benefit, of which the tax payers have to contribute more than 4 trillion dollars. Increasing the amortization contribution rate and speeding up amortization without addressing the benefit level improves the residual value of the fund at the end of our projection horizon, but it does not affect the contract value to the participants over our simulation horizon as their part of the contributions and their benefits are unchanged. The residual value of the fund measures the difference between its assets and the accrued liabilities to its participants at the end of the evaluation horizon. We do not assign the residual value to anyone in particular, but if it accrues to the tax payers, then faster amortization simply implies a shift in value from current to future tax payers.

A doubling of the contribution rate by the participants does affect their net benefit from the pension contract. The cohorts that have not yet entered the labor force lose about 70% of the value of their pension contract, while older participants lose about 13%. In money terms, the tax payers benefit by an amount of almost 1.5 trillion dollars, or about 30% and 40% of the contract value for young and older tax payers, respectively. Reducing benefits by cutting indexation can be quite effective too. Halving the indexation to CPI inflation reduces the contract value by 16% to 12% for young and old participants, respectively. Making indexation conditional on the funding ratio is also effective, leading to a reduction in the older participants' contract value by around 11%. These indexation policies benefit older tax payers by 4-6% of their contract value and young tax payers by about triple those figures. Changes in the pension fund's investment portfolio do not affect the participants values, again because their contributions and benefits are unchanged, though they do lead to shifts in value between tax payers and the fund's residual value.

The remainder of this paper is structured as follows. Section 2 relates this paper to the relevant literature. Section 3 presents the model, including a description of the demography, the pension fund and the calculation of the fund's benefits, contributions and liabilities. Sections 4 and 5, respectively, describe our economic scenario generator and the valuation of the cash flows following from the pension contract. Sections 6 and 7 summarize the data (sources) and the baseline settings, respectively, while Section 8 presents the various reforms we will consider.

Section 9 discusses the simulation results for the classic ALM and the value-based ALM. Finally, Section 10 concludes the main text of this paper. The appendices are not intended for publication, but will be posted on the authors' websites.

2   Relationship with the literature

A recent overview by (Munnell et al., 2013) of the funding ratios (ratios of assets over liabilities) of U.S. state pension funds illustrates the urgency of their financial condition. For a sample of 109 state plans and 17 locally-administered plans, the paper estimates the aggregate funding ratio for 2012 at 73%. Almost a quarter of the plans have a funding ratio below 60%, while only a small fraction of 6% have a funding ratio above 100%. The funding ratios are calculated on the basis of GASB standards that prescribe that assets are reported on an actuarially-smoothed basis, while the discount rate for the liabilities is typically set at around 8 percent, reflecting the expected long-term investment return on the fund's assets. These standards have been criticized by economists (Novy-Marx and Rauh (2009) and Bader and Gold (2003)) who claim that the future streams of benefit payments should be discounted at a discount rate reflecting their riskiness. As the state pension benefits are protected under most state laws, these payments can be seen as guaranteed and so this would plead for discounting them against the risk-free interest rate. Doing so would lead to a severe fall in the already-low average funding ratios of the state plans. Nevertheless, a precise assessment of the riskiness of state pension promises likely remains elusive for the foreseeable future. For example, the recent default of the city of Detroit may be an instructive case of whether pensions should be seen as a contractual obligation that cannot be diminished or impaired, or whether the holders of pension rights should be treated like other creditors, so that benefit cuts cannot be ruled out. In fact, a U.S. bankruptcy judge declared in December 2013 that legally pensions can be cut (Bomey and Priddle, 2013).

The value-based ALM approach that we apply in this paper to U.S. state pension funds has its roots in the pioneering papers of Sharpe (1976) and Sharpe and Treynor (1977) in utilizing derivative pricing to value contingent claims within pension funds. More recent applications of derivative pricing to pension plans are, among others, Blake (1998), Exley et al. (1997), Chapman et al. (2001), Ponds (2003), Bader and Gold (2007) and Hoevenaars and Ponds (2008). Our paper is closest in spirit to Biggs (2010), which also employs an option-based approach to value the market price of pension liabilities of U.S. state pension plans. However, it does not address reform plans and the associated redistribution effects. Value-based ALM has also been employed in the Dutch discussion on the redesign of the second-pillar pension funds offering DB plans. In particular, the CPB Netherlands Bureau for Economic Policy Analysis (2012) has applied it to investigate the generational fairness of various pension reform plans.

Our paper is complementary to related contributions studying the degree of underfunding of U.S. state pension plans, the causes of the underfunding and the measures that could be taken

to solve plan deficits. Novy-Marx and Rauh (2011) conduct a careful analysis of the value of the pension promises made by U.S. state pension funds under different assumptions about the riskiness of state pension plans and their seniority relative to state debt in the case of default. Several papers explore the extent of underfunding and the question what should be the appropriate target for the funding ratio (compare D'Arcy et al. (1999), Lucas and Zeldes (2009) and Bohn (2011)). Brown et al. (2011) argue that adequate funding requires 100% funding of the liabilities.

Regarding the causes of the underfunding of state pension plans some papers point at the implications of what may be called the 'accounting game'. The accountability horizon of politicians and public sector union leaders is much shorter than the horizon over which pension promises have to be met by adequate funding. Hence, union leaders and politicians tend to downplay the long-term costs of pension promises in favor of higher short-term wages and benefits (Mitchel and Smith (1994) and Schieber (2011)), as well as higher state spending in the short run. Shnitser (2013) stresses the role of the institutional design of pension plans in explaining the large variation in funding discipline. Institutional rules facilitating transparency and pre-commitment to expert-based financial and actuarial advice, in particular, appear to be conducive to mitigating underfunding and limiting the shift of costs to future tax payers. Kelley (2014) and Elder and Wagner (2014) apply a public choice perspective to explain underfunding issues.

Essentially, the literature has explored three broad directions of solutions to the underfunding problem: higher contributions, lower benefits and higher investment returns. Novy-Marx and Rauh (2014b) explore by how much contributions need to be raised to reach full funding in 30 years time. They compute a necessary increase of the order of 2.5 times the prevailing contribution level. The option of reducing benefits has long been seen as virtually impossible, as in many states public pensions are interpreted as hard contractual obligations, protected by civil law and state constitutions (Monahan, 2012). However, several states have already enacted benefit cuts and other measures to scale back the generosity of pensions. Further, as documented by Munnell et al. (2014), in response to the economic and financial crisis, many public plan sponsors have reduced or eliminated cost-of-living-adjustments in one way or the other. Munnell et al. (2014) report that between 2010 and 2013 this has been done by 17 states and conclude that benefits are no longer as secure as they were perceived to be before the financial crisis. Novy-Marx and Rauh (2014a) explore performance-linked indexation rules comparable to the one in the Wisconsin Retirement System or the conditional indexation rule in the Netherlands (Ponds and van Riel (2009) and Beetsma and Bucciol (2010, 2011a and 2011b)). Shnitser (2013) stresses that simply scaling back generosity or imposing higher contribution rates will not be enough. She claims that foremost institutional design has to be reframed to practices and rules that have proven to be successful in promoting funding discipline. A third route proposed to solve funding deficits is to raise expected investment returns. This option is already being exploited by U.S. state pension funds, as many of them

have high portfolio shares in equity, substantially higher than of pension funds outside the U.S. (Andonov et al. (2013) and OECD (2011)). However, finance-based papers (Black (1989), Peskin (2001), Bader and Gold (2003), Lucas and Zeldes (2009) and Pennacchi and Rastad (2011)) recommend that the strategic asset allocation should be set in line with the perspective of the tax payers as the party bearing the residual risk, which suggests to limit mismatch risk between the funds' assets and the market value of their liabilities. This may imply no equity investment at all when one aims at full protection of accrued benefits in the short run. Lucas and Zeldes (2009), employing a long-term perspective, suggest that the optimal portfolio should include some equity holdings as future pension liabilities are related to future wage growth and this growth is to some extent correlated with equity returns.

3   The model

This section describes first the demography, then the wage developments and, finally, the pension fund. Regarding the latter, we present the calculation of the so-called "actuarial assets", and the fund's liabilities vis-à-vis the retired and the workers. The "actuarial assets" and the total liabilities determine the funding ratio, which measures the fund's financial health, and they form inputs for the calculation of the contributions to the fund. When the fund's assets get fully depleted, the tax payer (the "sponsor") has to step in to ensure fulfillment of the pension contract.

3.1   The population

The participant population of the pension fund consists of individuals of ages 25 to 99 years. Individuals enter the fund at the age of 25 and remain with the fund for the rest of their life. Further, we assume that they retire at age a_R. The number of male and female participants of age a at time t is denoted as M_t^a and F_t^a respectively, where $a \in [25, 99]$. Using projections of survival probabilities we can calculate the size of these cohorts in the future. Concretely, $M_{t+n}^{a+n} = q_{a,t}^{m,n} M_t^a$ and $F_{t+n}^{a+n} = q_{a,t}^{f,n} F_t^a$, where $q_{a,t}^{m,n}$ and $q_{a,t}^{f,n}$ are the probabilities that a male, respectively female, person aged a in period t will survive another n years. The survival probabilities are deterministic. Hence, there is no longevity risk.

3.2   Wages

Crucial for the calculation of the pension liabilities are the wage developments. The wage level of the cohort of age a at time t is W_t^a and it is updated each time period. We do not take into account idiosyncratic wage risk, hence we assume a uniform wage level within each cohort. The wage levels across cohorts will be set to follow a certain career profile. Because we do not

have gender-specific age-wage profiles, we set the wage levels of males and females equal. The nominal wage level evolves as follows:

$$W_t^a = W_{t-1}^{a-1} w_{t-1}^{a-1},$$

where w_{t-1}^{a-1} is the gross wage growth rate from period $t-1$ to t for a cohort aged $a-1$ in period $t-1$. This factor is the product of the economy-wide gross wage growth rate w_{t-1} and a component $\tilde{w}_{t-1}^{a-1}$ attributable to the progression of the individual career:

$$w_{t-1}^{a-1} = w_{t-1} \tilde{w}_{t-1}^{a-1}. \tag{1}$$

We refer to $\tilde{w}_{t-1}^{a-1}$ as the promotion rate from period $t-1$ to t for someone aged $a-1$ in period $t-1$. Note that $\tilde{w}_{t-1}^{a-1}$ is always positive if the career profile has an upward sloping shape. We assume that the career profile remains constant over time, which implies that $\tilde{w}_{t-1}^{a-1} = \tilde{w}^{a-1}$ in all periods. Hence, the ratio of the wage levels of two workers of different ages is constant over time. The economy-wide wage growth rate w_{t-1} is stochastic and will be modeled as explained in Section 4 below.

3.3 The pension fund

State pension funds can differ in many respects. Some of the differences are parametric, such as the value of the discount rate to be applied when calculating the liabilities, while other differences are more fundamental. We model a pension fund based on the most common features across the entire population of pension funds available to us from the Public Plans Database of the Center for Retirement Research at Boston College (2014).

3.3.1 Assets

Market value of assets The market value of the fund's assets at the beginning of the next year, A_{t+1}, is equal to the asset value A_t at the beginning of this year, multiplied by its gross rate of return R_t, plus the net money inflow times the gross return over the half year over which it is on average invested:

$$A_{t+1} = A_t R_t + (C_t - B_t) R_t^{1/2}, \tag{2}$$

where C_t is the total amount of contributions received (calculated below) and B_t the total amount of benefits paid out. Since the benefits and contributions are (usually) paid on a monthly basis, while our model runs on a yearly basis, we assume that the payment of the benefits and contributions takes place in the middle of the calendar year and, hence, the net money inflow is invested on average for half a year until the beginning of next year.

Actuarial assets Pension fund assets in the U.S. are not measured at their market value when they are used as an input for pension policy. Rather, pension funds in the U.S. apply a certain smoothing procedure to come up with an actuarial value A_t^{act} of their assets in period t. Its detailed calculation is laid out in Appendix A. In short, the actuarial value of the assets at the end of the current year is equal to the actuarial value at the end of previous year, plus the net cash flows into the fund, plus the projected return on the assets, and a recognition of the smoothed excess of actual above expected investment income. In the special case that initial actuarial assets equal the initial market value of the assets, $A_0^{act} = A_0$, and the smoothing period is shrunk to a single period, one has that $A_t^{act} = A_t$, for all $t \geq 0$. Hence, in this specific case the process for actuarial assets coincides with that of the market value of the assets.

3.3.2 Calculation of the liabilities

In this section we will formalize the calculation of the liabilities, closely following Munnell et al. (2008b) and Novy-Marx and Rauh (2011). Total liabilities L_t are calculated as the present value of the projected future benefit payments to all current pension fund participants, taking into account the survival probabilities. The group of participants comprises the employees and the retired. The total liabilities are the sum of the liabilities L_t^m and L_t^f to the male and female participants. The liabilities to each gender, in turn, are calculated by multiplying the individual liability $L_t^{a,\zeta}$ to an age-a and gender-ζ individual by the number of gender-ζ individuals in this cohort and then summing over the cohorts. Hence, the total liabilities are calculated as:

$$
L_t = L_t^m + L_t^f = \sum_{a=25}^{99} \left(M_t^a L_t^{a,m} + F_t^a L_t^{a,f} \right),
$$

$$
L_t^{a,\zeta} = \sum_{i=\max(a_R-a,0)}^{99-a} \left(\tilde{R}_t^{(i)} \right)^{-i} q_{a,t}^{\zeta,i} K_{t,t+i}^{a+i},
$$

(3)

where $K_{t,t+i}^{a+i}$ is the projection at time t of the pension pay-out i years ahead for a participant of age a and $\tilde{R}_t^{(i)}$ is the gross interest rate that the *pension fund* uses to discount to period t the cash flows materializing i periods into the future. It will be based on the median discount rate used by the pension funds in our dataset. We notice that $\tilde{R}_t^{(0)} = 1$ and $q_{a,t}^{\zeta,0} = 1$. U.S. public pension plans usually use a discount rate that is flat over the entire projection horizon and equal to the expected asset portfolio return. Summarizing, the liabilities to an individual of a particular cohort depend on the years they will (still) receive benefits, the level of the benefits, the discount rate and the survival probabilities. In the following we first explain the calculation of the (projected) benefits of the pensioners and then the calculation of the projected benefits of the workers.

Calculation of pensioners' benefits To protect the living standards of the retired as much as possible, pensioners receive a cost-of-living adjustment (COLA), which we denote as π_t. The

COLA usually covers a certain share of inflation and it is often capped at a predetermined level (Center for Retirement Research at Boston College, 2014):

$$\pi_t = 1 + \min\{\alpha \cdot cpi, cap\},$$

where cpi stands for consumer-price inflation, α is the fraction of inflation compensated and cap is the maximum indexation rate. (Both cpi and cap are expressed as a fraction of unity.)

The current pay-out to a retiree equals the current pension rights, B_t^a, while the projected pay-out i periods from now equals the current pension rights adjusted for expected future indexation:[1]

$$K_{t,t}^a = B_t^a, \forall a \in [a_R, 99],$$

$$K_{t,t+i}^{a+i} = B_t^a \prod_{j=1}^{i} \pi_{t+j}^p, \forall i \in [1, 99-a], \forall a \in [a_R, 98],$$

where π_{t+j}^p is the projected COLA based on the average annual inflation projection by the pension funds in our dataset. Current pension rights B_t^a equal the pension rights at the moment of retirement increased by the past indexation since then:

$$B_t^a = B_{t-(a-a_R)}^{a_R} \prod_{j=0}^{a-(a_R+1)} \pi_{t-j}, \forall a \in [a_R+1, 99].$$

For example, the pension rights of a 70-year old person, who retired at the age $a_R = 65$, are equal to the benefit calculated when that person reached the age of 65, plus the indexation that has been awarded over the 5 years since then.

A common feature of the state pension plans in the U.S. is that they are based on the average wage preceding the moment of exit from the workforce. Concretely, the benefit of somebody retiring in year $t - (a - a_R)$ can be calculated as the product of the accrual rate ε, the number of years in the workforce (here, 40) and the average wage level over the past z years:

$$B_{t-(a-a_R)}^{a_R} = \frac{40\varepsilon}{z} \sum_{l=1}^{z} W_{t-(a-a_R)-l}^{a_R-l}, \forall a \in [a_R, 99].$$

The averaging period z varies from one to five years, with the majority of public plans applying a three–year average (Munnell et al., 2012).

Because the pension rights for the retired are already determined and, hence, cannot be affected

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[1]In practice, COLAs are only included in the calculation of the liabilities to the pensioners if there exists an explicit policy rule on how the indexation is awarded. COLAs are not taken into account if indexation is determined on an ad-hoc basis.

by future service or salary increases, they are independent of the method we use to recognize the liabilities – see below.

Calculation of workers' projected benefits   The calculation of the projected pay-outs to current employees is similar to that of the pensioners, i.e. as the pension rights B_t^a at time t of someone of age a, adjusted for future indexation during retirement:[2]

$$K_{t,t+i}^{a+i} = B_t^a, i = a_R - a, \forall a \in [25, a_R - 1],$$

$$K_{t,t+i}^{a+i} = B_t^a \prod_{j=a_R+1-a}^{i} \pi_{t+j}^p, \forall i \in [a_R + 1 - a, 99 - a], \forall a \in [25, a_R - 1],$$

where the first line gives the projected benefit during the first year of retirement, while the second line gives the projected benefit during the ensuing years in retirement, which takes into account the projected COLAs to be received during retirement. Determining the pension rights B_t^a is not as straightforward for the working population as it is for the pensioners. Unlike pensioners, the active participants continue working and, hence, they are expected to accrue more years of service and (in most cases) to have a salary at retirement date that is higher than their current salary. Hence, depending on the method used, their pension rights B_t^a can be recognized in various ways that we will discuss now.

Methods for recognizing liabilities   There are different methods to recognize liabilities. Here, we will explain the so-called *Accrued Benefit Obligation* (ABO) method, the *Projected Benefit Obligation* (PBO) method and the *Projected Value of Benefits* (PVB) method.[3] The latter is also known as the Present Value of Future Benefits method.

Under the *ABO method*, only the pension rights accrued until time t are taken into account. Hence, B_t^a is based on the number of years since entry into the workforce and the average of at most the past z wage levels:

$$B_t^{25} = 0,$$

$$B_t^a = \frac{(a - 25)\varepsilon}{\min(a - 25, z)} \sum_{l=1}^{\min(a-25,z)} W_{t-l}^{a-l}, \forall a \in [26, a_R].$$

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[2] No indexation is given for the years before retirement.

[3] There is also the Vested Benefit Obligation (VBO), which considers only employees who have served at least the fund's vesting period, if there is one. However, this variant is not meaningful in our context, because we assume that everybody works for 40 years at the same public sector employer.

The youngest cohort of age 25 has just entered the fund and has no rights accrued yet. The rights of the other young cohorts who do not yet have z years in service are based on the average of the available wage history. For the cohorts that have at least z years of service, the pension rights are the product of the years in the workforce, the accrual rate and the average pay over the past z years. Hence, for an individual worker the ABO pension rights increase with each additional year of service.

Under the *PBO method*, we also take into account the effect of expected future salary increases on the rights accrued up to now. Hence, under this method B_t^a is the projected benefit level at retirement when expected future salary advances are taken into account:

$$B_t^{25} = 0,$$
$$B_t^a = \frac{(a-25)\varepsilon}{z} \sum_{l=1}^{z} \mathrm{E}_t \left[W_{t+(a_R-a)-l}^{a_R-l} \right], \forall a \in [26, a_R - 1].$$

Again, the youngest cohort, the 25 years old, has no accrual yet and, hence, it has no pension rights in terms of the PBO. However, for a given age and assuming that the nominal wage level is expected to increase, the pension rights for the other working cohorts are higher under the PBO method than under the ABO method. When the worker nears her retirement age, the difference between accrued rights in PBO and ABO terms shrinks.

Finally, state civil service jobs are relatively secure, so that the pension fund might in addition consider the rights that the employees will acquire in the future if they continue working in their job until retirement. The *PVB method* takes this into account. Therefore, it defines the pension rights B_t^a including future accrual due to new service:

$$B_t^a = \frac{40\varepsilon}{z} \sum_{l=1}^{z} \mathrm{E}_t \left[W_{t+(a_R-a)-l}^{a_R-l} \right], \forall a \in [25, a_R - 1]. \tag{4}$$

Hence, this measure is based on the accrual over a full working life. Note that in this case the generation that has just entered the labor force already has a stake in the pension liabilities, as opposed to ABO and PBO. At any given age before retirement rights computed according to the PVB method exceed rights computed according to the ABO or PBO method, because the PVB method takes into account both expected future increases in the wage level and expected future accrual. All three measures merge for a given individual at the moment of her retirement.

3.4 The funding ratio

The evaluation of the pension fund's financial position is usually based on its funding ratio (FR), which we calculate here as the ratio of its actuarial assets over its liabilities:

$$FR_t = \frac{A_t^{act}}{L_t}.$$

The funding ratio measures the financial health of the pension fund and is an important quantity to guide the policies of the fund.

3.5 Total benefits

The total amount of pension benefits to be paid out in year t is:

$$B_t = \sum_{a=a_R}^{99} (M_t^a + F_t^a) B_t^a.$$

3.6 Contributions

The value of the additional pension rights earned in a given year is called the *normal cost*. The present value of the future normal costs will be used to calculate what is called the fund's *accrued liability*, which will be compared with the fund's actuarial assets to establish the required amortization payments.

3.6.1 The entry-age normal costing method

The most common method for calculating the normal cost in public plans is the so-called *entry-age normal costing* (EAN) method. Under the EAN method, the employer's annual normal cost associated with an individual participant is calculated as a payment throughout the projected years of service needed to finance the PVB obligation. Due to salary growth pension rights increase more than linearly over time. Hence, the method implies a component of front-loading, because the employer is pre-paying some of the future accrual (Munnell et al., 2008b).

The so-called *normal cost rate* (NCR) of an active participant is calculated at the entry age as the ratio of the present value of future benefits and the present value of career salary levels:

$$NCR_t^{25,\zeta} = \frac{L_t^{25,\zeta}}{PVW_t^{25,\zeta}}, \quad \zeta \in \{f; m\},$$

where $L_t^{25,\zeta}$ is the liability to someone of gender ζ who enters the labor force in period t as calculated in (3) on the basis of the PVB method, i.e. based on pension rights as calculated in

(4), and $PVW_t^{25,\zeta}$ is the present value of all future wages throughout the participant's career as projected at entry. Hence, the normal cost rate is the percentage payment of a worker's projected career salary needed to cover the cost of the projected pension benefits for that worker.

Further, the present value of future wages of a worker of age a and gender ζ is calculated as:

$$PVW_t^{a,\zeta} = \sum_{i=0}^{a_R-1-a} \left(\tilde{R}_t^{(i)}\right)^{-i} q_{a,t}^{\zeta,i} \cdot \mathrm{E}_t\left[W_{t+i}^{a+i}\right],$$

which takes account of the survival proabilities.

The *present value of the future normal cost* ($PVFNC$) of a worker are the normal costs that will be recognized throughout her remaining years of service:

$$PVFNC_t^{a,\zeta} = PVW_t^{a,\zeta} \times NCR_{t-(a-25)}^{25,\zeta}.$$

Hence, this is the product of the present value of future wages and the normal cost rate determined at the time when the worker entered the fund.

3.6.2 The accrued liability

The so-called *accrued liability* (L_{accr}) of a participant is the difference between the liabilities to this participant, calculated as the present value of the projected benefits, and the present value of the future normal cost:

$$L_{accr,t}^{a,\zeta} = L_t^{a,\zeta} - PVFNC_t^{a,\zeta}.$$

If we follow an individual worker over time, we will see that the present value of the future normal cost will decrease, as there will be fewer remaining years to pay the normal cost. Therefore, the accrued liability associated with an individual worker increases over time. The accrued liability of pensioners is simply equal to the present value of their future benefits, since they should have paid the whole normal cost before reaching the retirement age. Therefore,

$$PVFNC_t^{a,\zeta} = 0 \Rightarrow L_{accr,t}^{a,\zeta} = L_t^{a,\zeta}, \forall a \in [a_R, 99].$$

Finally, the fund's accrued liability is the sum of the individual accrued liabilities over genders and cohorts:

$$L_{accr,t} = L_{accr,t}^m + L_{accr,t}^f = \sum_{a=25}^{99} \left(M_t^a L_{accr,t}^{a,m} + F_t^a L_{accr,t}^{a,f}\right). \tag{5}$$

3.6.3 Calculation of the contributions

The annual contribution to the pension fund shared is between the employer and the workers. It is the sum of two components: the normal cost and the amortization payment of the so-called *unfunded actuarial accrued liability* ($UAAL$), defined as the difference between the fund's accrued liability and the actuarial value of the assets:

$$UAAL_t = L_{accr,t} - A_t^{act}.$$

We assume a smoothing period of u years for the amortization of the $UAAL$, implying that the required amortization payment in period t is:

$$AMORT_t = \begin{cases} \frac{1}{u}UAAL_t & \text{if } UAAL_t \geq 0, \\ 0 & \text{if } UAAL_t < 0, \end{cases}$$

Hence, the amortization payment cannot be negative. Therefore, if there is a funding surplus, the total contribution is kept at the normal cost covering contribution. Munnell et al. (2008a) show that only in about half of all the pension plans the annually required contribution is paid. In our simulations we introduce a parameter λ that captures the fraction of the required amortization payment that is *actually* paid.

The aggregate volume of contributions by all the participants and the employer in year t is:

$$C_t = \sum_{a=25}^{a_R-1} \left(NCR_{t-(a-25)}^{25,m} M_t^a W_t^a + NCR_{t-(a-25)}^{25,f} F_t^a W_t^a \right) + \lambda AMORT_t.$$

The total contribution rate c_t is expressed as a percentage of the total wage sum in year t:

$$c_t = \frac{C_t}{\sum_{a=25}^{a_R-1}(M_t^a + F_t^a)W_t^a}. \tag{6}$$

It can be split into a contribution rate c_t^W paid by the worker and a contribution rate c_t^T paid by the employer (i.e., the tax payer):

$$c_t = c_t^W + c_t^T.$$

Typically, the worker pays a fixed contribution rate, in which case c_t^W is constant, while the employer pays the remaining part. Hence, in our simulations we will assume a constant worker contribution rate.

3.7   Sponsor support

Forecasting the development of a pension fund over a long horizon implies that there exist scenarios in which the fund's assets get depleted. Therefore, we assume that, whenever the fund's assets become insufficient to finance the net cash outflow, the employer, i.e. the tax payer, provides *sponsor support* by replenishing the fund's assets.[4] The sponsor support, SS_t, comes on top of the tax payer's regular contribution:

$$SS_t = \begin{cases} 0 & \text{if } A_t \geq B_t - C_t, \\ (B_t - C_t) - A_t & \text{if } A_t < B_t - C_t, \end{cases}$$

so that the sponsor support covers the difference between the assets and the net cash outflow $B_t - C_t$. Once the sponsor support becomes positive, the fund's assets are depleted and the fund effectively continues to be run on a pay-as-you-go basis. The sponsor support is included in the calculation of the actuarial assets (see Appendix A). We define the sponsor support contribution rate as:

$$cSS_t = \frac{SS_t}{\sum_{a=25}^{a_R-1}(M_t^a + F_t^a)W_t^a}. \tag{7}$$

4   The economic scenario generator

We estimate a quarterly vector autoregression (VAR) model on historical data for the U.S. and use this model to generate a set of economic scenarios. It is generally not known which scenario will occur in reality. Hence, we evaluate the performance of an object of interest under all the scenarios that we generate. For our purposes, there is no need for a very refined model specification, hence, we simply use a first-order VAR model linking the state vector in quarter $q+1$ to that in quarter q:

$$X_{q+1} = \begin{bmatrix} y_{q+1} \\ xs_{q+1} \\ cpi_{q+1} \\ w_{q+1} \end{bmatrix} = \alpha + \Gamma X_q + \varepsilon_{q+1}, \tag{8}$$

where y_{q+1} is the short-term quarterly interest rate, xs_{q+1} is the excess return on stocks, cpi_{q+1} is consumer price inflation and w_{q+1} is real wage growth. The first two variables are calculated by taking the natural logarithm of the gross rate. The latter two variables are calculated as changes in the logarithm of the relevant index. The noise term ε_{q+1} follows a multivariate

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[4]Henceforth, we will refer to the employer as the "tax payer".

normal distribution:

$$\varepsilon_{q+1} \sim \mathbb{N}(0, \Sigma). \tag{9}$$

Further,

$$\alpha = (\mathrm{I} - \Gamma)\mu, \tag{10}$$

where I denotes the identity matrix and μ is the unconditional mean of the vector X_q for all q. Equation (9) implies that:

$$\mathrm{E}_q\left[X_{q+1}\right] = (\mathrm{I} - \Gamma)\mu + \Gamma X_q. \tag{11}$$

The estimates $\hat{\alpha}$ and $\hat{\Gamma}$ for α and Γ, respectively, are obtained by estimating equation (8) using OLS. We obtain $\hat{\Sigma}$ as the variance-covariance estimate of the vector of error terms. Given $\hat{\Gamma}$ and $\hat{\alpha}$, the estimate $\hat{\mu}$ is:

$$\hat{\mu} = (\mathrm{I} - \hat{\Gamma})^{-1}\hat{\alpha}. \tag{12}$$

The estimates $\hat{\alpha}$, $\hat{\Gamma}$, $\hat{\Sigma}$ and $\hat{\mu}$ are used for pricing the cash flows.

5 Valuation

A pension plan can be seen as a financial contract. If we perceive this contract as a combination of contingent claims, we can value the pension deal using the derivative pricing techniques of risk-neutral valuation introduced by Black and Scholes (1973). Specific valuation issues of contingent claims in pension contracts are discussed in Hoevenaars and Ponds (2008), Lekniute (2011) and Ponds and Lekniute (2011).

To value the contract we use the scenarios for the underlying state vector produced by our scenario generator. One can calculate the market value of a contingent claim by using deflators and real-world scenarios (the "P world"), or using risk-neutral scenarios (the "Q world"). Both alternatives should result in the same outcome. Here, we will use risk-neutral valuation. We aim at valuing the pension contract to the different stakeholders of the fund. Therefore, we will calculate the value of the net benefits to each cohort of fund participants and each cohort of tax payers. The annual cash flows associated with the pension plan are generated on the basis of the paths of the vector of state variables obtained through risk-neutral sampling. We will value the cash flows over a horizon of T years. Recall that the scenarios are simulated at the quarterly frequency, while the cash flows take place at the annual frequency.

The value $V_0^{P,a}$ of the pension contract to participants of the cohort aged a at $t = 0$ is:

$$V_0^{P,a} = \mathrm{E}_0^Q\left[(R_1^f)^{-0.5}NB_1^a + \sum_{t=2}^{T}\left(\prod_{j=1}^{t-1}(R_j^f)^{-1}\right)(R_t^f)^{-0.5}NB_t^a + \left(\prod_{j=1}^{T}(R_j^f)^{-1}\right)L_{accr,T}^a\right], \tag{13}$$

where E_0^Q is the risk-neutral expectation under the Q measure of the cash flows discounted against the gross risk-free rate R_j^f. Here, R_j^f is based on the gross return on a short-term risk-free bond and calculated by cumulating the quarterly short rates obtained through the simulation of the state vector. Further, NB_t^a is the net benefit, which for workers is their contribution and for pensioners is their benefit payment, hence NB_t^a is negative for workers and positive for retirees. Because NB_t^a occurs in the middle of time period t, it is discounted only for half of period t. Since the working participants continue accruing pension rights in exchange for their contributions, we also add the final accrued liabilities to the total value of the contract for each cohort. This means we define the pension contract value to a cohort as the contributions they have paid, the benefits they have received, and pension rights accrued in exchange for those contributions.

We define the total value to all participants as the sum of the values for each cohort participating in the fund during the evaluation horizon:

$$V_0^P = \sum_a V_0^{P,a},$$

Further, we calculate the value of the contract to the tax payers as

$$V_0^T = - \mathrm{E}_0^Q \left[(R_1^f)^{-0.5} \left(C_1^T + SS_1 \right) + \sum_{t=2}^T \left(\prod_{j=1}^{t-1} (R_j^f)^{-1} \right) (R_t^f)^{-0.5} \left(C_t^T + SS_t \right) \right]. \qquad (14)$$

where C_t^T is the tax-payers' contribution in dollars, i.e. the normal cost plus the actual amortization payment minus the contribution by the employees. Finally, at the end of the evaluation horizon the pension fund may have some assets left, of which the value is calculated as:

$$V_0^R = \mathrm{E}_0^Q \left[\prod_{j=1}^T (R_j^f)^{-1} (A_T - L_{accr,T}) \right]. \qquad (15)$$

Since the final assets in the fund are equal to the initial assets plus all the money inflows and all the money outflows, we can write

$$A_0 = V_0^R + V_0^T + V_0^P. \qquad (16)$$

Appendix B shows the details for the pricing of the cash flows and the transformation to the risk-neutral scenarios needed to calculate the value of the pension contract to the various stakeholders.

6 The data

All our data are for the U.S. Our dataset comprises macroeconomic data, data from financial markets, data on state pension funds and demographic data. The economic scenario generator described earlier is based on historical data spanning the third quarter of 1971 up to and including the last quarter of 2012, except for the real yields which are only available starting from 2003. We use historical time series of the short interest rate, stock returns, price inflation and wage inflation. The short interest rate is the 3-month Treasury rate series and is obtained from two sources. The data until 1982 come from the Handbook of Monetary Economics (McCulloch, 1990). The data from 1982 onwards is available through the Economic Data from the Federal Reserve Bank of St. Louis (2014a). For the stock returns we use the value-weighted NYSE/Amex/Nasdaq return including dividends, which is available from the Center for Research in Security Prices (2014b). For price inflation we use the CPI index provided in the Treasury and Inflation Indices dataset by the Center for Research in Security Prices (2014a). Finally, wage inflation is the compensation of employees, wages and salary accruals, retrieved from the Economic Database of Federal Reserve Bank of St. Louis (2014b). These data are all quarterly or of higher frequency and transformed into quarterly data. To value the cash flows we use a term structure of interest rates based on U.S. Treasury zero-coupon rates for maturities from 1 to 10 years from the Federal Reserve Board (2014). The treasury real yield curve rates for maturities 5, 7 and 10 are obtained from US Department of the treasury (2014). Pension plan data are obtained from the Public Plans Database of the Center for Retirement Research at Boston College (2014). As far as the demography is concerned, we use the National Population Estimates for 2010 by single year of age and sex provided by the United States Census Bureau (2014). Further, we use the survival rates derived from the Cohort Life Tables for the Social Security Area by Year of Birth and Sex provided by the U.S. Social Security Administration (2014). These life tables are available for birth years with 10 years intervals. Therefore, we linearly interpolate the survival rates for the birth years in between the ones directly available.

7 Baseline settings for our U.S. state plan

The annual averages in our scenario set and used for further calculations are 11.5% for the stock returns, 6.6% for 10-year Treasury bonds, 5.7% for nominal wage growth and 3.7% for consumer price inflation.

We set the simulation horizon to 75 years, which is quite a common horizon for official program projections. Hence, we evaluate the pension contract for cohorts born up to fifty years from now, while we assume that for cohorts born later contributions are set such that their net contract value is zero. Extending the evaluation horizon does not seem very meaningful, because of the substantial economic and institutional uncertainties over the very long run.

At the start $t = 0$ of the simulation horizon we set the population of our pension fund equal to the aggregate number of participants in 2010 of all the state plans covered by the data from the Center for Retirement Research at Boston College (2014). As a result, our pension fund initially has approximately 24 million participants. While these data do not include all the U.S. public pension plans, they do include a very substantial share of the U.S. public sector workers at the sub-national level. Hence, the magnitude of the net burden on the tax payers that we will expose below provides a good indication of the seriousness of the public sector pension underfunding problem for the U.S., although it is likely only a lower bound to the problem. The initial demographic structure of the fund is assumed to be identical to that of the U.S. population in 2010. Hence, the relative sizes of the cohorts are equal to those for the U.S. in 2010.

The fund's demography evolves as follows over the simulation horizon. Throughout the simulation horizon new cohorts aged 25 enter the labor market and become participants of the pension fund. Hence, for the first 25 years of the simulation horizon we determine the sizes of the 25-year old cohort by taking the sizes of the cohorts younger than 25 in the U.S. population in 2010 and projecting the size of each such cohort when it reaches the age of 25 by using the officially projected (gender-specific) survival probabilities. The number of new entrants for the remaining 50 years of a simulation run is calculated by extending over time the trend in the size of the 25-year old cohort, i.e. by calculating the average annual growth rate of the 25-year old cohort over the first 25 years of the simulation horizon and taking this as the annual growth rate for the next 50 years. At the start of the simulation there are about 260 thousand 25-year old males and about 250 thousand 25-year old females entering the fund. During the first 25 years the number of new 25-year olds decreases on average by 0.34% per year. Extrapolating this trend over the next 50 years yields roughly 200 thousand and 190 thousand new male and female entrants, respectively, in the final simulation year.

Our baseline simulation is based on the most common characteristics of the U.S. state pension plans. Its liabilities are computed by using the entry age normal actuarial method for liability recognition and are based on the following actuarial assumptions. Individuals retire at the age of 65. Hence, a full career means that they work for 40 years. The career (age-wage) profile is obtained from Bucciol (2012). Because the shape of this profile does not change over time, the wage earned at a given age increases each year with the common wage growth rate in the economy. The $t = 0$ average wage rate is set such that the fund's initial liabilities L_0 are equal to 3,400 billion dollars, which is the aggregate amount of liabilities in 2010 over all pension funds in the Public Plans Database of the Center for Retirement Research at Boston College (2014).[5] The benefit factor, or accrual rate, is 2% for each additional year of service, while the earnings base for the retirement benefit is the average of the final three years of pay during

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[5]Effectively, this implies that we have to scale down the wage relative to the actual average U.S. wage in 2010. Implicitly, this rescaling corrects for the potential presence of vesting periods, which are not explicitly included in our model of the pension fund, and the difference between average state sector and economy-wide wages.

the career. During retirement there is always full indexation for consumer price inflation, also when inflation is negative.

Projected benefits are heightened up with a projected salary increase of 5.4% for each working year and a projected annual rate of price inflation of 3.5% for each year in retirement. These projections are based on the average projections of salary increases and price inflation applied by the funds in our sample. Future retirement benefits are discounted at a fixed rate of 8% a year, which is the current median for the U.S. state pension funds. Hence, the implicit assumption is that the fund's assets earn an average annual return of 8%. We assume that the initial accrued pension rights, B_0^a, to have been fully indexed up to time $t = 0$.

Contributions to the fund are calculated as follows. The annually required contribution is set to the normal cost plus the required amortization payment, which we assumed to be fixed at zero in the case of a fund surplus (one-sided policy). The normal cost is calculated as a percentage of the projected career salary based on the EAN actuarial method. The amortization payment is determined by spreading the unfunded actuarial accrued liability $UAAL$ in equal annual payments over the next 30 years, with a moving 30-years window. Hence, we use the so-called *open amortization period*. Employees pay a fixed 6% contribution of their salary, while the tax payer as employer pays the remainder of the required contribution. Our sample data show that, assuming that the normal cost is paid in full, the actual amortization payment is on average 10% of the required amortization payment. However, these data are for the year 2010, the year following the year with the worst economic performance since the Great Depression and a year with unusually large declines in the value of stocks. This suggests that the share of the required amortization payment paid in that year may have been unusually low. Because of the lack of potentially more representative data we assume that the actual contribution is set to 100% of the normal cost, plus 50% of the required amortization payment, i.e. $\lambda = 0.5$. However, below we will also examine variations on the baseline in which less or more of the required amortization payment is paid in a year.

We set the initial funding ratio FR_0 equal to 75%, which is close to the average 2010 funding ratio of 73% of the pension funds in our database. Based on FR_0 and the initial liabilities, which are thus both matched to our data, we obtain the initial actuarial assets of the fund:

$$A_0^{act} = FR_0 \cdot L_0.$$

The funding position of the pension fund in the subsequent periods is calculated as the ratio of the actuarial assets and the amount of liabilities.[6]

We assume that the fund invests 50% of its assets in fixed income and the other 50% in risky assets. This implies an average annual return on the fund's portfolio of 9%. We thus abstract from investments in assets other than these two classes.

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[6]The computation of the actuarial assets makes use of a 5-year smoothing period of realized investment income in excess of expected investment income – see Appendix A.

8 The reforms

We consider a number of variations on the baseline plan, which we refer to as Plan 0.0, to explore what various policy changes imply for the contract values of the various stakeholders. This can give us leads about the effectiveness of different measures in increasing the financial sustainability of the pension fund and, hence, in reducing reliance on the support from the tax payers.

We consider three groups of measures, which we summarize in Table 1. The first set of measures, Plans 1.1-1.4, addresses the contribution rate. Plans 1.1 and 1.2 vary the fraction of the required amortization payment actually paid, Plan 1.3 shortens the period over which the amortization payment is spread, while Plan 1.4 doubles the contribution payment by the participants from 6% to 12%, but leaves the total contribution rate unchanged. In other words, in Plan 1.4 the financial health of the pension fund itself is unaffected, while the burden on the tax payers is alleviated.

Table 1: Summary of the alternative plans

Plan	Description
0.0	Baseline
Contribution	
1.1	0% amortization paid ($\lambda = 0$)
1.2	100% amortization paid ($\lambda = 1$)
1.3	Amortization spread over 10 years
1.4	Participants' contribution rate doubled to 12%
Indexation	
2.1	Indexation at 0.5 cpi
2.2	Conditional indexation
Portfolio composition	
3.1	100% stocks
3.2	0% stocks

The second set of alternatives addresses the degree of indexation. As under the baseline plan, under each alternative plan, when CPI inflation is negative, indexation is always full (thus negative). Plan 2.1 halves indexation when CPI inflation is positive. Under Plan 2.2, if CPI inflation is positive, then indexation to CPI inflation is conditional on the level of the funding ratio FR. Specifically, if $cpi \geq 0$, indexation is 0, if $FR < 0.5$, and $(2FR - 1)cpi$, if $FR \geq 0.5$. In other words, if CPI inflation is positive, indexation is zero for funding ratios below one half, while it increases linearly with the funding ratio for funding ratios of at least one half. Hence, a funding ratio above unity implies more than full indexation. This way of providing conditional indexation is closely related to the way most Dutch pension funds index their pension rights, (see Ponds and van Riel (2009) and Beetsma and Bucciol (2011b)). We continue to set contributions on the basis of a normal cost that is still calculated under the assumption of full indexation. We

do this in order to see the isolated effect of a change in indexation. Otherwise, the comparison with the baseline would be contaminated by a fall of the normal cost when indexation is reduced, which through lower contributions would in turn dampen the beneficial effect of less indexation on the fund's financial health.

The third group of measures concerns the composition of the fund's asset portfolio, which we vary from zero to 100% stocks.

9   Results

9.1   The "classic" ALM results

This subsection discusses the "classic" (i.e., not applying market-based valuation) ALM simulation results under the baseline and the alternative policies. Here, and in the sequel, we simulate a set of 5000 economic scenarios each over a horizon of 75 years. This horizon allows us to take into account a large number of cohorts that enter the fund after the start of the evaluation horizon. Moreover, it is a commonly-used horizon for pension policy evaluation in the U.S.

Table 2a reports for the various plans the 5%, 50% and 95% percentiles after 25 years for the funding ratio (FR), the pension result (PR), the total contribution rate (c), the normal cost payment (cNC), the amortization payment ($cAmort$) and the sponsor support payment (cSS). The pension result is defined as the ratio of cumulative granted indexation to cumulative price inflation, i.e. $PR_t = \prod_{\tau=0}^{t} (1 + \pi_\tau) / \prod_{\tau=0}^{t} (1 + cpi_\tau)$. Hence, the lower it is, the larger the deterioration in the purchasing power of the benefits. All the contribution measures are expressed in percentages of the wage sum. Table 2b reports all the corresponding numbers after 75 years.

Consider first the results for the baseline variant 0.0. After 75 years the median value of the funding ratio is at 96%, only somewhat above its initial value. However, we observe a wide spread in the development of the funding ratio. Its fifth percentile after 75 years is 0, implying that the fund is effectively run on a pay-as-you-go basis, with benefits on a period-by-period basis paid from participants' and tax payers' contributions. The substantial probability of deterioration of the funding ratio is not surprising in view of the fact that only 50% of the required amortization costs is paid when the fund's financial health is poor. The pension result is by construction equal to unity, because full indexation is always granted, including a negative adjustment in the case of deflation. Table 3 reports the probability that all assets are fully depleted within the 75-year simulation horizon. This probability is 7% for the baseline, while, conditional on a full depletion taking place, the median year in which this happens is 65 years from the start of the simulation.

Table 2: Classic ALM results

(a) after 25 years

	0.0	**1.1**	**1.2**	**1.3**	**1.4**	**2.1**	**2.2**	**3.1**	**3.2**
FR 5%	34%	17%	45%	54%	34%	42%	47%	21%	27%
FR 50%	74%	61%	83%	89%	74%	86%	84%	103%	39%
FR 95%	167%	158%	175%	181%	167%	188%	153%	486%	59%
PR 5%	100%	100%	100%	100%	100%	53%	43%	100%	100%
PR 50%	100%	100%	100%	100%	100%	67%	72%	100%	100%
PR 95%	100%	100%	100%	100%	100%	80%	132%	100%	100%
c 5%	16%	16%	16%	16%	16%	16%	16%	16%	22%
c 50%	20%	16%	20%	20%	20%	18%	18%	16%	24%
c 95%	25%	16%	30%	33%	25%	23%	22%	26%	25%
cAmort 5%	0%	0%	0%	0%	0%	0%	0%	0%	5%
cAmort 50%	3%	0%	4%	3%	3%	1%	2%	0%	7%
cAmort 95%	8%	0%	13%	17%	8%	7%	6%	10%	9%
cNC	16%	16%	16%	16%	16%	16%	16%	16%	16%
cSS 5%	0%	0%	0%	0%	0%	0%	0%	0%	0%
cSS 50%	0%	0%	0%	0%	0%	0%	0%	0%	0%
cSS 95%	0%	0%	0%	0%	0%	0%	0%	0%	0%

(b) after 75 years

	0.0	**1.1**	**1.2**	**1.3**	**1.4**	**2.1**	**2.2**	**3.1**	**3.2**
FR 5%	0%	0%	44%	62%	0%	39%	52%	0%	0%
FR 50%	96%	0%	145%	175%	96%	199%	115%	297%	0%
FR 95%	797%	688%	896%	964%	797%	1105%	242%	8144%	8%
PR 5%	100%	100%	100%	100%	100%	17%	12%	100%	100%
PR 50%	100%	100%	100%	100%	100%	26%	85%	100%	100%
PR 95%	100%	100%	100%	100%	100%	37%	2515%	100%	100%
c 5%	18%	18%	18%	18%	18%	18%	18%	18%	31%
c 50%	18%	18%	18%	18%	18%	18%	18%	18%	32%
c 95%	32%	18%	34%	34%	32%	26%	24%	32%	33%
cAmort 5%	0%	0%	0%	0%	0%	0%	0%	0%	13%
cAmort 50%	1%	0%	0%	0%	1%	0%	0%	0%	14%
cAmort 95%	14%	0%	16%	16%	14%	8%	6%	14%	15%
cNC	18%	18%	18%	18%	18%	18%	18%	18%	18%
cSS 5%	0%	0%	0%	0%	0%	0%	0%	0%	0%
cSS 50%	0%	15%	0%	0%	0%	0%	0%	0%	6%
cSS 95%	6%	25%	0%	0%	6%	0%	0%	8%	11%

Note: classic ALM results for the 5%, 50% and 95% percentiles of the funding ratio (FR), the pension result (PR), the total contribution rate (c), which is split into amortization ($cAmort$) and normal cost (cNC), and sponsor support payment (SS) for the base Plan 0.0 and alternative contribution (Plans 1.1-1.4), indexation (Plans 2.1-2.2) and investment (Plans 3.1-3.2) policies.

Table 3: Likelihood of full depletion of assets

	0.0	1.1	1.2	1.3	1.4	2.1	2.2	3.1	3.2
Probability	7.2%	51.8%	0.0%	0.0%	7.2%	0.1%	0.0%	13.5%	76.4%
Year, 5%	46	29	-	-	46	62	-	32	59
Year, 50%	65	48	-	-	65	67	-	55	68
Year, 95%	74	71	-	-	74	71	-	73	74

Note: the first line reports the probability that the fund's assets are fully depleted within the 75-year simulation horizon. The following lines show the quantiles for the distribution of the years of depletion, conditional on scenarios in which depletion takes place within the simulation horizon. As an example, under the baseline Plan 0.0, conditional on depletion taking place, 45 is the maximum number of years in which this happens in less than 5% of the cases. Therefore, the quantile is attained at 46.

We now discuss the variations on the baseline setting. Under all the alternatives the component of the contribution rate based on the normal cost increases from approximately 16% after 25 years to somewhat less than 18% after 75 years. When no amortization payments are made (Plan 1.1), the total contribution rate is equal to the normal cost. However, this leads to a dramatic expected deterioration of the funding ratio. Even the median funding ratio after 75 years is 0 and the chances that the fund's assets are fully depleted within the simulation horizon rises to 51.8%. By contrast, a shift to 100% amortization under Plan 1.2 tends to produce a gradual improvement of the funding ratio; the median funding ratio after 75 years is 145%. The spread in the distribution of the funding ratio is still quite high, but extremely low funding ratios are not very likely at the end of the horizon. Indeed, the probability of a full depletion of the fund's assets within the simulation horizon is zero. The flip side of the improvement in the financial position of the fund is, of course, that the total contribution rate c rises. The median contribution rate after 25 years is slightly less than 1 percentage point above the median under the baseline and 4 percentage points above the median under Plan 1.1. In the long run, however, the improved financial position of the fund allows contribution rates to fall again. After 75 years, no amortization payment is needed anymore under Plan 1.2 on average, whereas the base plan with only half of required amortization paid still requires 1%-point amortization on average. Reducing the amortization period to 10 years, Plan 1.3, has a similar and, in fact, even stronger positive effect on the fund's financial position. Compared to Plan 1.2 the distribution of funding ratios after 75 years shifts even further upwards. Of course, in the meantime the total contribution rate tends to be even higher, with a ninety-fifth percentile of 33% after 25 years, as compared to 30% under Plan 1.2 and 25% under the base plan. However, at the end of the full horizon, the median amortization contribution rate has fallen to zero and, hence, the total contribution rate is 1%-point below the level under the baseline Plan 0.0. The reported numbers for funding ratios, total contributions and amortization contributions under Plan 1.4 are identical to those under the baseline. However, the component of the normal cost paid by the participants has been doubled.

The second set of alternative measures introduces changes in the indexation rate. Plan 2.1 always yields lower indexation when CPI inflation is positive, implying an expected improvement in the financial position of the funding over the long run. Although halving the indexation rate has a substantial effect on the purchasing power of the pension benefits, the improvement in the funding ratio may not be as spectacular as one might a priori expect, because the total contribution in the meantime falls below that under the baseline. While the normal cost rate remains the same, the amortization contribution rate becomes lower than under the baseline. After 25 years, the median total contribution rate is 2 percentage points lower than under the baseline. At the end of the full horizon, the median amortization contribution rate has fallen to zero and the difference with the baseline in the median of the total contribution rate is 1 percentage point.

A shift from the baseline to conditional indexation (Plan 2.2) also tends to improve the financial position of the fund. Compared to Plan 2.1, where indexation is halved, the spread in the distribution of the funding ratios becomes smaller, with the fifth percentile under Plan 2.2 at the end of the 75-year horizon located at 52% instead of 39% under Plan 2.1. The compression in the distribution of the funding ratios is not surprising, because the policy is specifically aimed at stabilizing the funding ratio with smaller indexation given when the fund's financial situation is unhealthy and higher, possibly more than full, indexation given when the fund runs a surplus. As a result, the distribution of the total contribution rates is also more compressed. At the end of the full horizon, the median amortization contribution rate is zero.

Our third set of measures looks at changes in the pension fund's asset portfolio. Not surprisingly, a policy of investing 100% of the portfolio in stocks, Plan 3.1, raises the median funding ratio compared to the baseline, but it also raises the spread in funding ratios. After 75 years there is a non-negligible probability that the fund's assets are fully depleted. A complete shift out of stocks, Plan 3.2, leads over time to a downward shift of the entire distribution of the funding ratios, such that even the ninety-fifth percentile of the distribution after 75 years is only 8% under this plan. This is not surprising, as the normal cost is based on liabilities that are calculated with a discount rate higher than the expected portfolio return under Plan 3.2, implying a systematic shortage of resources in the fund.

Overall, it can be concluded that under many of the settings studied here the pension plan is likely to be unsustainable. The factors contributing to this conclusion are the fund's initial underfunding and the long amortization period in combination with the actual amortization contribution rate being lower than the required rate. A policy of reducing the indexation rate helps in keeping the plan financially sustainable, although such a policy may not be as effective as is a priori expected. Also a policy of conditional indexation helps in keeping the fund financially healthy. The advantage of such a policy is that it compresses the distribution of the funding ratios, because the indexation rate is explicitly linked to the funding ratio itself.

9.2 The value-based ALM results

The classic ALM results discussed above have provided us with a number of interesting results regarding the effectiveness of alternative policies to strengthen the financial sustainability of our pension fund. Value-based ALM is useful to assess changes in the market value of the pension contract to its various stakeholders when shifting from the baseline plan to another plan. To this end, it is instructive to show the balance sheet in Table 4. The left-hand side reports the value of the initial assets of the pension fund (A_0) and the present value of the contributions plus sponsor support paid by all cohorts of tax payers (V_0^T) over the 75 years evaluation period. On the right-hand side we find the present value of the net benefits aggregated over all cohorts of participants (V_0^P). These net benefits consist of all benefits received minus the contributions paid by the participants over the evaluation horizon plus the (PVB-based) *accrued* liabilities to those still alive at the end of the evaluation horizon. The latter are the liabilities to the retired at the end of the horizon, plus the liabilities to the workers at the end of the horizon, minus the present value of the future normal cost. What remains is the present value of the fund residue (V_0^R), which is the difference between the present value of the final assets at the end of the evaluation horizon minus the present value of the accrued liabilities.[7]

Table 4: The balance sheet

A_0	V_0^P
V_0^T	
	V_0^R

We evaluate at $t = 0$ the baseline contract (denoted by a tilde) and changes relative to the baseline. Reforms are zero-sum games in value terms across the funds' stakeholders, as follows from (16):

$$\Delta V_0^P + \Delta V_0^T + \Delta V_0^R = 0,$$

where V_0 is the value of the new plan and $\Delta V_0 \equiv V_0 - \tilde{V}_0$ the change in value from the baseline. For example, if the contributions of the tax payers to the pension fund increase, this implies that the value of the fund residue, the value of the net benefits to the participants, or both, must increase. The initial assets from the balance sheet do not appear in this expression, as we start with the same value of initial assets under all alternative plans. Hence, the change in the value of the initial assets is always 0. We are also interested in the relative change ΔRV_0 in the values of the various stakeholders, computed as:

$$\Delta RV_0 = \frac{V_0 - \tilde{V}_0}{|\tilde{V}_0|} \times 100\%.$$

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[7]Alternatively, the residual value could have been attributed to the tax payers, since it is net of the accrued liabilities to the participants alive at the end of the evaluation horizon. However, in the sequel we keep it separate from the tax payers' value based on their contributions and the sponsor support.

We report the results in a number of ways. In particular, we report values for the entire group of participants, the entire group of tax payers, individual cohorts of participants, individual cohorts of tax payers, and for the residue of the fund. The value of the participant cohort aged a in period 0 is denoted by $\tilde{V}_0^{P,a}$ or $V_0^{P,a}$ and follows directly from the cash flows associated with the pension contract. The aggregate cash flows in a year from all the tax payers together are immediately available. However, calculating the values $\tilde{V}_0^{T,a}$ or $V_0^{T,a}$ for the individual cohorts of tax payers aged a in period 0 requires an assumption about the allocation of the aggregate cash flows across the cohorts of tax payers. We assume that the demographic structure of the tax payer population is identical to that of the participant population and that the cash flows assigned to the individual tax payer cohorts are proportional to the shares of the individual cohorts in aggregate income. Hence, for each cohort of tax payers, the contribution is based on the size of the cohort and the income level of each individual cohort member. For tax payers of working age, the relative contribution of each cohort is fixed over time through the age-wage profile, which is constant over time. The relative contribution of each cohort of retired tax payers is based on the average (across all simulated data points for the set of scenarios) income under the baseline plan.

9.2.1 Baseline

Figure 1 shows for the baseline, Plan 0.0, the value of the contract $\tilde{V}_0^{P,a}$ and $\tilde{V}_0^{T,a}$ in billions of dollars for the various cohorts of fund participants and tax payers, respectively. The youngest generation is the one to be born in 50 years (age -50 at $t = 0$), whereas the oldest is 99 years old at the start of the simulation period. The figure gives an impression of the order of magnitude of the aggregate amounts at stake for the various cohorts of stakeholders in the state-level civil sector pension funds. In present value terms, under the baseline plan over the 75-year simulation horizon, the aggregate net benefit including the accrual over all participant cohorts is approximately 7 trillion dollars, while the aggregate contribution over all tax payer cohorts is approximately 4 trillion dollars.[8] This is a sizable number and, as indicated earlier, it likely forms a lower bound on the actual amount of redistribution from tax payers to fund participants. Importantly, also, the figure shows the relative distribution of the net benefits across the various participant cohorts and of the financing burden across the various cohorts of tax payers. It is useful to notice that the positive and negative areas in the figure do not sum to zero, because the value of the residue of the fund after 75 years generally differs from the value of the fund's initial assets. We see that the value of the contract is positive for all participant cohorts. This is not surprising for cohorts that have already been working for a while, i.e. the cohorts older than 25 at $t = 0$, because their contributions are sunk, while the benefits are still ahead of them. As of the age of around sixty, the contract value starts falling steeply with age. For those that are already retired, i.e. of age 65 and over at $t = 0$, the amount of future benefits

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[8]In our discussion of these numbers we ignore the fact that part of the tax payer population are also participants in the fund. Thus, we focus on the gross redistribution between the groups and not on the net redistribution, which would require more information to construct reliable numbers.

to be received is shrinking with age. In addition, older cohorts have been accumulating fewer rights, ceteris paribus, because the real wage levels were lower during their work career than for younger cohorts.
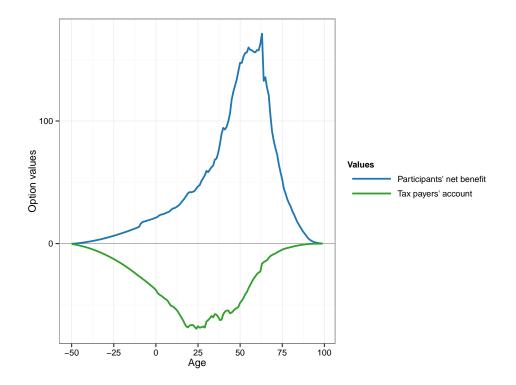
Figure 1: Stakeholders' option values under the base plan

Participants' net benefit and tax payers' values of the contract per age cohort under Plan 0.0, in billion dollars

For the cohorts of age 25 and younger at $t = 0$, the positive contract value needs mote explanation, because in a completely actuarially-neutral system, those who have not yet been contributing would experience zero value from their pension contract: the value of their future benefits must be offset by the value of their future contributions. However, this value is only zero for the cohort born at $t = 50$, because over the evaluation horizon this cohort will pay zero contributions and thus build up zero rights. For future generations that will be born before $t = 50$, the value of the benefits they will receive over the evaluation horizon plus the value of the accrued rights at the end of the evaluation period exceeds the value of the contributions to be made during the evaluation period. This net benefit can only be financed out of two sources: a reduction in the residual value $\tilde{V}_0^R$ of the fund or contributions from the tax payers. From the figure we observe that the value of the plan $\tilde{V}_0^{T,a}$ is negative for all cohorts of tax payers.

9.2.2 Outcomes of the reforms

We report changes in dollar values and in relative values for groups of cohorts of participants and tax payers resulting from reforms. The results are reported in Table 5a (for changes in dollar values) and 5b (for changes in relative values) for two groups of participants and two

groups of tax payers. Negative relative values denote a deterioration compared to the baseline for the particular stakeholders under consideration. The group of the young (superscript Y) comprises all the cohorts younger than 25 or yet unborn at $t = 0$, while the group of the old (superscript O) comprises all the cohorts of age 25 or older at $t = 0$, hence all the active or retired participants at $t = 0$. The table also reports changes in the value of the residue at the end of the evaluation horizon. As explained above, theoretically, for each alternative plan, the sum of value changes must be zero, i.e. $\Delta V_0^{P,Y} + \Delta V_0^{P,O} + \Delta V_0^{T,Y} + \Delta V_0^{T,O} + \Delta V_0^R = 0$. However, due to numerical inaccuracies small deviations from zero may be possible.

Under Plans 1.1-1.3 neither the contributions made by the plan participants nor the indexation rules are changed, so $\Delta V_0^{P,Y} = \Delta V_0^{P,O} = 0$. A reduction in the amortization payment (Plan 1.1) benefits current tax payers, because they see their amortization contribution rate fall. The value of the fund's residue falls, in line with the expected deterioration of the funding ratio over time. The policy change also has an effect on the future cohorts of tax payers, as they have to cover the deficit with sponsor support payments if assets get depleted completely. An increase in the amortization payment, Plan 1.2, implies higher contribution payments by the tax payers, so both young and old tax payers experience a loss of value. A similar conclusion holds for a shortening of the period over which the amortization takes place, Plan 1.3. The residue value improves, but at the cost of higher contributions by the current and future tax payers, who see the value of their stake in the pension arrangement fall by a total of roughly 0.2 and 0.7 trillion dollars, respectively. A doubling of the contribution rate by the participants, Plan 1.4, reduces the total contribution paid by the tax payers and, thus, shifts value from both groups of participants to the tax payers. The total value shift from the participants to the tax payers is more than 1.5 trillion dollars. In percentage terms the older participants lose 13% of the contract value, while the younger participants lose more than two thirds of their value. The financial position of the fund is unchanged throughout and, hence, the change in the residue value is zero.

Changes in indexation policy shift value between the participants and the tax payers. Halving indexation, Plan 2.1, lowers the plan value to the participants and raises the plan value to the tax payers, who have to pay smaller amortization and sponsor support contributions. The total value loss to the participants is about 0.9 trillion dollars, or 16% of the contract value of the younger participants and 12% of the contract value of the older participants, of which about 40% is shifted to the tax payers and the remainder to an increased residue value of the fund, because the funding ratio is expected to improve in the long-run (relative to the baseline), while the accrued pension rights of the participants at the end of the evaluation horizon are lower. Conditional indexation, Plan 2.2, produces qualitatively similar value shifts across the stakeholders, except for the young participants. Under Plan 2.2 no indexation is given in bad states of the world, at times when the funding ratio is low and it tends to be most valuable, and more of it is given in good states of the world, when indexation is less valuable. This leads to a 0.7 trillion dollar loss for the older participants, but improves the situation for the young

Table 5: Effects of plan changes on stakeholders

(a) Level effects

Plan	Description	$\Delta V_0^{P,Y}$	$\Delta V_0^{P,O}$	$\Delta V_0^{T,Y}$	$\Delta V_0^{T,O}$	ΔV_0^{R}
Contribution						
1.1	0% amortization paid	0	0	-103	466	-372
1.2	100% amortization paid	0	0	-152	-400	565
1.3	amortization 10 years	0	0	-216	-678	914
1.4	part. contr. rate doubled	-815	-733	689	859	0
Indexation						
2.1	indexation is 0.5 CPI	-188	-670	253	84	523
2.2	conditional indexation	44	-652	343	117	152
Portfolio composition						
3.1	100% stocks	0	0	-323	-118	501
3.2	0% stocks	0	0	180	37	-183

(b) Relative effects

Plan	Description	$\Delta RV_0^{P,Y}$	$\Delta RV_0^{P,O}$	$\Delta RV_0^{T,Y}$	$\Delta RV_0^{T,O}$	ΔRV_0^{R}
Contribution						
1.1	0% amortization paid	0%	0%	-5%	22%	-290%
1.2	100% amortization paid	0%	0%	-7%	-19%	440%
1.3	amortization 10 years	0%	0%	-10%	-32%	711%
1.4	part. contr. rate doubled	-69%	-13%	33%	41%	0%
Indexation						
2.1	indexation is 0.5 CPI	-16%	-12%	12%	4%	407%
2.2	conditional indexation	4%	-11%	16%	6%	119%
Portfolio composition						
3.1	100% stocks	0%	0%	-15%	-6%	390%
3.2	0% stocks	0%	0%	9%	2%	-143%

Note: the table reports the effects of switching from baseline Plan 0.0 to Plans 1.1-3.2 on future plan participants ($\Delta V_0^{P,Y}$, $\Delta RV_0^{P,Y}$), current plan participants ($\Delta V_0^{P,O}$, $\Delta RV_0^{P,O}$), future tax payers ($\Delta V_0^{T,Y}$, $\Delta RV_0^{T,Y}$), current tax payers ($\Delta V_0^{T,O}$, $\Delta RV_0^{T,O}$) and the residue of the pension fund (ΔV_0^{R}, ΔRV_0^{R}). Panel (a) reports changes in billions of dollars. Panel (b) reports relative changes as percentages of the baseline values. Negative numbers imply a deterioration of the value for that stakeholder.

participants who benefit from the improved funding position.

Our third set of changes concerns changes in the composition of the asset portfolio of the pension fund. In all of the previous plans the asset mix was kept constant at 50% fixed income and 50% risky assets. When the portfolio allocation is the same across the plans, the total amount of risk remains unchanged, but it is shifted among stakeholders due to policy changes, like changes in the contribution or indexation rules. However, changing the asset mix changes the risk of the pension fund. Under a symmetrical contract this should not lead to value transfers, as under such a contract higher volatility is rewarded with higher expected returns. However, the pension plan policy in question is not symmetrical from the tax payers' perspective. When bad returns materialize, tax payers have to cover the deficit by increased amortization and sponsor support payments. However, good returns do not necessarily lead to lower contribution payments as the contribution can never fall below the normal cost level. To sum up, the contribution payments are limited on the downside, but unlimited on the upside. When riskier investments are made (Plan 3.1) the upwards potential of the returns goes to the residue of the pension fund, but the downwards risk is allocated to the tax payers in the form of higher amortization and sponsor support payments. Therefore we see negative value changes for the tax payers and an increase in residue value. In the case of a derisking of the portfolio (Plan 3.2), the opposite effects can be seen. A much lower downside risk of portfolio returns results in a lower probability of high amortization and sponsor support payments, hence the tax payers are better off. However, the lower expected portfolio returns implied by the less risky asset mix affect the residue value negatively. Since neither the indexation policy, nor the participant contribution rules have changed, the fund participants are unaffected by changes in the composition of the fund's asset portfolio.

Figures 2 - 4 depict the value consequences of contract changes for all individual cohorts of participants and tax payers. In line with the figures reported in the above tables, we see indeed that moving from the baseline plan to Plan 1.1 with zero amortization the older tax payers are better off, as part of the contribution burden is shifted to the future, while younger cohorts of tax payers have to make up for the losses of the fund. Moving to 100% amortization, Plan 1.2, reduces the value for the older tax payers, but benefits the younger tax payers and ensures that the residual value of the fund is raised. Speeding up amortization, Plan 1.3, also affects most tax payer cohorts negatively over the simulation horizon, while doubling the contribution by the participants affects all participants' values negatively and all tax payers' values positively.

Not surprisingly, a shift from full indexation to compensation of only half of the CPI, i.e. a shift from Plan 0.0 to Plan 2.1, lowers the net benefit value for all cohorts currently alive and all cohorts born up to 10 years from the start of the simulation horizon. All cohorts that enter the fund later see no change in value, because they will not receive any pension benefit over the simulation horizon. Recall that under all plans that we consider, including Plan 2.1, the calculation of the contributions is done as if indexation is full. However, under this plan actual indexation in retirement is adjusted. Hence, over the simulation horizon cohorts that
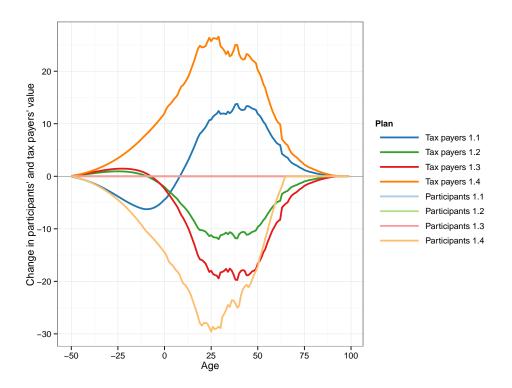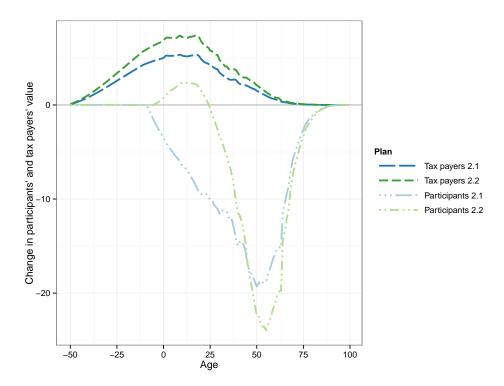
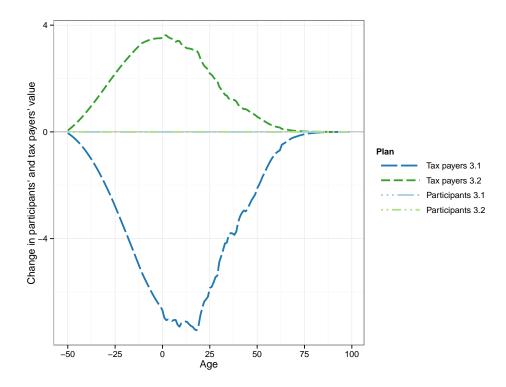Figure 2: Stakeholders' value changes due to contribution reforms

Change in stakeholder values (in billions of dollars) when policy is changed from the base Plan 0.0 to plans with no amortization payment (Plan 1.1), full amortization payment (Plan 1.2), 10 years amortization smoothing (Plan 1.3) and double contribution rate by employees (Plan 1.4).

Figure 3: Stakeholders' value changes due to indexation reforms

Change in stakeholder values (in billions of dollars) when policy is changed from the base Plan 0.0 to plans with half CPI indexation (Plan 2.1) and conditional indexation (Plan 2.2).

Figure 4: Stakeholders' value changes due to investment strategy reforms

Change in stakeholder values (in billions of dollars) when policy is changed from the base Plan 0.0 to plans with 100% risky assets (Plan 3.1) and 100% fixed income assets (Plan 3.2).

do not reach retirement within the simulation horizon are not affected by this plan. Since the change in indexation does affect benefit payments, it affects the final value of the assets and thus the residual value of the fund. The final liabilities are thus affected only through the part of the liabilities that belongs to the final pensioners, as this part incorporates all actual indexation received since retirement. Given that inflation is mostly positive, getting only half of the inflation indexed leads to lower pension benefit payments and consequently a lower value of the contract. Similarly, since indexation tends on average to be lower under conditional indexation, also Plan 2.2 affects the value for most participants in a negative way; only the young ones benefit slightly. The flip side is that all the tax payer cohorts benefit from the two plan changes.

Shifts in the portfolio composition of the pension fund are completely neutral in value terms for all the participants, because their benefits and contribution payments are unaffected, while we see changes in the value for the tax payers. All tax payer cohorts suffer from a move to a 100% equity portfolio, while they benefit from a complete move out of equity.

10   Concluding remarks

This paper has explored the financial sustainability and redistributive aspects of a typical U.S. state DB pension fund under unchanged policies. The results confirm what is generally feared,

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namely that current pension policies are unlikely to be financially sustainable. Therefore, we explored a number of alternative policies, involving changes in the composition of the investment portfolio of the pension fund, the structure of pension contributions and policies aimed at reducing benefits, in particular policies that reduce indexation. Some of these measures can indeed be quite effective at improving the fund's financial position. Indexation conditional on the funding ratio is promising, because it can improve the fund's financial position, while keeping the potential spread in the funding ratio limited. The reason is that indexation rises with the size of the funding ratio and may even become more than full for large funding ratios.

We have also applied a market-based valuation approach to the cash flows generated by the fund and we found that under the baseline plan all the cohorts of participants experience substantial net benefits at the expense of all the tax payer cohorts. Policies that raise the contributions by the participants or that reduce the indexation of benefits are quite effective at shifting part of the participants' net benefits back to the tax payers. We estimate the effect of halving the indexation to CPI inflation at a contract value reduction of around 15% for the participants. Conditional indexation also lowers most participants' contract value and increases the residue. In dollar terms, the alleviation of the burden on the tax payers is on the order of 0.3 - 0.5 trillion dollars for the changes in indexation policy we consider, while a doubling of the participants' contribution rate produces an alleviation of the tax payers' burden by around 1.5 trillion dollars.

The sustainability and redistribution issues exposed by this paper may well be a lower bound to the true extent of the potential future financial difficulties that the U.S. civil servants' pension arrangements face. First, our dataset did not include all civil servants' pension funds. A priori, there is little reason to assume that the funds that have not been included are financially healthier than the funds that were part of our dataset. Second, since we did not want to take a specific stand on the future equity premium, we have worked under the assumption that the future stock returns are in line with those in the past. However, if future equity premia turn out to be lower than those in the past, a scenario that many would consider quite likely, state pension funds' financial problems may become substantially more severe than suggested in the present paper.

A rather general conclusion from our analysis is that the substantial net benefit of the promises to fund participants at the cost of tax payers could be an argument for fund boards and participants to initiate steps towards improving their funding by raising contributions or by cutting back on benefit levels. The looming financial burden on tax payers and the prospective deterioration of public services may discourage individuals of working age from working in states where the pension funding problems are worst. An outcome with falling spending on public goods and rising taxes merely paid to finance civil servants' retirement schemes may become politically unsustainable and result in states defaulting. This could well cause larger value losses to fund participants than those associated with orderly reforms backed by a majority of the population.

The analysis in this paper can be extended into various directions, for example by exploring the consequences of closing the fund for new participants or closing it even for contributions from current participants. Also, by applying our method of value-based asset-liability management, we can study value shifts between tax payers and participants for individual state funds and explore for which states pension reform is most urgent. Obviously, there are many other potential applications of value-based asset-liability management in the area of pension reform.

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Appendices

Not for publication – for referees' use

A   Calculation of the actuarial assets

Define the Investment Income Amount Of Immediate Recognition ($IIAOIR$) as a target investment income based on the *expected* return $\bar{R}_t$:

$$IIAOIR_t = A_t(\bar{R}_t - 1) + (C_t + SS_t - B_t)(\bar{R}_t^{1/2} - 1). \tag{17}$$

The so-called Investment Income Market Total ($IIMT$) denotes the actual realization of investment income, i.e. the difference between the market value of the assets at the end of the year and the market value of the assets at the beginning of the year, less the net cash inflow associated with contributions and benefits. From equation (2) we get:

$$IIMT_t = A_{t+1} - A_t - (C_t + SS_t - B_t) = A_t(R_t - 1) + (C_t + SS_t - B_t)(R_t^{1/2} - 1). \tag{18}$$

Finally, the Investment Income Amount For Phased In Recognition ($IIAFPIR$) is realized investment income in excess of expected investment income:

$$IIAFPIR_t = IIMT_t - IIAOIR_t.$$

Hence, $IIAFPIR_t$ is positive when the actual investment return exceeds its expected value, and vice versa. The usual smoothing procedure to calculate actuarial assets involves taking the average of the excess investment incomes over the past. We define the Total Recognized Investment Gain ($TRIG$) as the average of $IIAFPIR$ over a smoothing horizon of v years:

$$TRIG_t = \frac{1}{v}\sum_{i=0}^{v-1} IIAFPIR_{t-i}.$$

Then, the actuarial value of the assets at the beginning of year $t+1$ will be determined by raising the actuarial value of the assets at the beginning of year t with the net money inflows (contribution payments minus the benefit pay-outs), investment income of immediate recognition and the smoothed value of the excess investment income over the past v years:

$$A_{t+1}^{act} = A_t^{act} + (C_t + SS_t - B_t) + IIAOIR_t + TRIG_t. \tag{19}$$

B   Simulation procedure

This appendix provides the details on the simulation of both the classic ALM and the value-based ALM model. In both cases we draw a set economic scenarios. Each scenario involves drawing of a path of 300 quarters of our state vector. Under the classic ALM we generate those paths drawing shocks from a zero-mean normal density function, while under value-based ALM we generate these paths using risk-neutral sampling, thereby effectively drawing shocks from a normal density function with a negative mean. The path for the state vector thus gives us a path for the stock returns. Also, for each quarter into the simulation, we determine the term structure using an affine model based on the state variables. This allows us to calculate the return on the fixed income part of the fund's portfolio, so that we now have the returns on the entire fund portfolio. Using the path of the state vector generated under the risk-neutral sampling, as well as the returns on the fund's portfolio based this same path of the state vector, allows us to generate the annual cash flows associated with the pension contract under the value-based ALM. These are then discounted against the risk-free rate of interest, as explained in the main text.

Below we first discuss the general pricing framework, followed by the calculation of the fund's portfolio returns. Finally, we discuss the risk-neutral sampling procedure used for the risk-neutral valuation.

B.1   General pricing framework

In line with the literature (Campbell et al., 1996), we assume that the stochastic discount rate $-m_{q+1}$ for the real-world scenarios is given by the following function of the state vector generated by our VAR model and the shocks to this state vector:

$$-m_{q+1} = e_y' X_q + \frac{1}{2}(\beta_0 + \beta_1 X_q)' \Sigma (\beta_0 + \beta_1 X_q) + (\beta_0 + \beta_1 X_q)' \varepsilon_{q+1}, \tag{20}$$

where β_0 and β_1 are a vector and a matrix of parameters respectively and e_y indicates the position of the short rate in the state vector:

$$e_y = (1, 0, 0, 0)'.$$

Consider some derivative (e.g., the pension contract) with a payoff of Z_τ at time τ, which is a function of the path $X_1, X_2, ... X_\tau$ of the state vector. The price of the derivative at time q is then given by

$$P_q = \sum_{\tau=q+1}^{\infty} \mathrm{E}_q \left[Z_\tau \exp \sum_{s=q+1}^{\tau} m_s \right].$$

In a complete market it is possible to sell the derivative at time $q+1$ for its price P_{q+1}. Hence the following must hold:

$$P_q = \mathrm{E}_q \left[P_{q+1} \exp m_{q+1} \right], \tag{21}$$

where P_{q+1} is the total price based on the total return index where any payoff is reinvested in the same index. If we use lower-case letters to denote log-values so that

$$p_q = \log P_q,$$

knowing that X_{q+1} has a Gaussian distribution, and using the properties of the log-normal distribution, we can derive from equation (21):

$$\exp p_q = \mathrm{E}_q \left[\exp p_{q+1} \exp m_{q+1} \right] = \mathrm{E}_q \left[\exp(p_{q+1} + m_{q+1}) \right]$$
$$= \exp \left(\mathrm{E}_q \left[p_{q+1} + m_{q+1} \right] + 1/2 \, \mathrm{Var}_q \left[p_{q+1} + m_{q+1} \right] \right),$$

Hence,

$$p_q = \mathrm{E}_q \left[p_{q+1} + m_{q+1} \right] + \tfrac{1}{2} \, \mathrm{Var}_q \left[p_{q+1} + m_{q+1} \right]. \tag{22}$$

Note that

$$\mathrm{E}_q \left[m_{q+1} \right] = -e_y' X_q - \tfrac{1}{2} (\beta_0 + \beta_1 X_q)' \Sigma (\beta_0 + \beta_1 X_q), \tag{23}$$

$$\mathrm{Var}_q \left[m_{q+1} \right] = (\beta_0 + \beta_1 X_q)' \Sigma (\beta_0 + \beta_1 X_q). \tag{24}$$

We will now apply our pricing framework to the various assets that are relevant for us.

B.1.1 Nominal bonds

Denote by $p_q^{(n)}$ the quarter-q log price of a zero coupon nominal bond that matures at time $q + n$ and pays one unit of currency at maturity date. We assume that it is an affine function of the state variables:

$$p_q^{(n)} = -D_n - H_n' X_q. \tag{25}$$

Its nominal yield $Y_q^{(n)}$ satisfies the following relationship:

$$\left(1 + Y_q^{(n)} \right)^{-n} = P_q^n. \tag{26}$$

Denote $y_q^{(n)} \equiv \ln(1 + Y_q^{(n)})$. Hence, equations (25) and (26) imply

$$y_q^{(n)} = -\frac{1}{n} p_q^{(n)} = \frac{1}{n} D_n + \frac{1}{n} H_n' X_q. \tag{27}$$

Applying the general pricing equation (22) to zero-coupon bonds and using equation (25) we get:

$$
\begin{aligned}
p_q^{(n)} &= \mathrm{E}_q\left[p_{q+1}^{(n-1)} + m_{q+1}\right] + 1/2\,\mathrm{Var}_q\left[p_{q+1}^{(n-1)} + m_{q+1}\right] \\
&= -D_{n-1} - H'_{n-1}\,\mathrm{E}_q\left[X_{q+1}\right] \\
&\quad + \mathrm{E}_q\left[m_{q+1}\right] \\
&\quad + \frac{1}{2}Var_q\left[-H'_{n-1}X_{q+1}\right] \\
&\quad + \frac{1}{2}\,\mathrm{Var}_q\left[m_{q+1}\right] \\
&\quad + \mathrm{Cov}_q\left[-H'_{n-1}X_{q+1}, m_{q+1}\right].
\end{aligned} \tag{28}
$$

Using (23), (24) and (11) we can rewrite (28) as

$$
\begin{aligned}
p_q^{(n)} &= -D_{n-1} - H'_{n-1}\left((I-\Gamma)\mu + \Gamma X_q\right) \\
&\quad - e'_y X_q - \frac{1}{2}(\beta_0 + \beta_1 X_q)'\Sigma(\beta_0 + \beta_1 X_q) \\
&\quad + \frac{1}{2}\left(H'_{n-1}\Sigma H_{n-1}\right) \\
&\quad + \frac{1}{2}(\beta_0 + \beta_1 X_q)'\Sigma(\beta_0 + \beta_1 X_q) \\
&\quad + H'_{n-1}\Sigma(\beta_0 + \beta_1 X_q),
\end{aligned}
$$

where the last part follows from

$$
\begin{aligned}
\mathrm{Cov}_q\left[-H'_{n-1}X_{q+1}, m_{q+1}\right] &= \mathrm{Cov}_q[-H'_{n-1}\left((I-\Gamma)\mu + \Gamma X_q + \varepsilon_{q+1}\right), \\
&\quad - e'_y X_q - \frac{1}{2}(\beta_0 + \beta_1 X_q)'\Sigma(\beta_0 + \beta_1 X_q) - (\beta_0 + \beta_1 X_q)'\varepsilon_{q+1}] \\
&= \mathrm{E}_q\left[(-H'_{n-1}\varepsilon_{q+1})(-(\beta_0 + \beta_1 X_q)'\varepsilon_{q+1})\right] \\
&= \mathrm{E}_q\left[(H'_{n-1}\varepsilon_{q+1})(\varepsilon'_{q+1}(\beta_0 + \beta_1 X_q))\right] \\
&= H'_{n-1}\,\mathrm{E}_q\left[\varepsilon_{q+1}\varepsilon'_{q+1}\right](\beta_0 + \beta_1 X_q) \\
&= H'_{n-1}\Sigma(\beta_0 + \beta_1 X_q).
\end{aligned}
$$

We can further rewrite $p_q^{(n)}$ as

$$
\begin{aligned}
p_q^{(n)} &= -D_{n-1} - H'_{n-1}(I-\Gamma)\mu - H'_{n-1}\Gamma X_q \\
&\quad - e'_y X_q \\
&\quad + \frac{1}{2}H'_{n-1}\Sigma H_{n-1} \\
&\quad + H'_{n-1}\Sigma\beta_0 + H'_{n-1}\Sigma\beta_1 X_q \\
&= -D_{n-1} - H'_{n-1}(I-\Gamma)\mu + \frac{1}{2}H'_{n-1}\Sigma H_{n-1} + H'_{n-1}\Sigma\beta_0 \\
&\quad - (e'_y + H'_{n-1}\Gamma - H'_{n-1}\Sigma\beta_1)X_q.
\end{aligned}
$$

The last equation is already of the affine structure as in equation (25) with parameters

$$D_n = D_{n-1} + H'_{n-1}(I - \Gamma)\mu - \frac{1}{2}H'_{n-1}\Sigma H_{n-1} - H'_{n-1}\Sigma\beta_0,$$

$$H_n = e_y + (\Gamma - \Sigma\beta_1)'H_{n-1}. \tag{29}$$

For $n = 0$ we have that:

$$p_q^{(0)} = \ln P_q^{(0)} = \ln 1 = 0,$$

which is given by equation (25) with parameters

$$D_0 = 0,$$

$$H_0 = 0. \tag{30}$$

Using (30) in (29) we get

$$D_1 = 0,$$

$$H_1 = e_y. \tag{31}$$

This implies

$$p_q^{(1)} = -e'_y X_q = -y_q^{(1)}. \tag{32}$$

The deflator is calibrated on the short rate so that this constraint is satisfied.

B.1.2   Real bonds

Denote by $P_{s,q}^{r(n)}$ the price of a real bond at time q issued at time $s \leq q$ and maturing at time $q + n$. Such a bond pays $\frac{\Pi_{q+n}}{\Pi_s}$ at maturity, where Π_q is the price index at time q. This implies that:

$$P_{q-1,q}^{r(n)} = \frac{\Pi_q}{\Pi_{q-1}}P_{q,q}^{r(n)}, \tag{33}$$

which can be expressed in terms of logarithms as:

$$p_{q-1,q}^{r(n)} = \pi_q + p_{q,q}^{r(n)}. \tag{34}$$

At maturity the nominal payoff of real bond is equal to the inflation during the bonds life, so in real terms the payoff is equal to one. The real n-period yield is thus:

$$Y_q^{r(n)} = \left(\frac{1}{P_{q,q}^{r(n)}}\right)^{\frac{1}{n}}, \tag{35}$$

which in logarithmic terms is:

$$y_q^{r(n)} = -\frac{1}{n}p_{q,q}^{r(n)}. \tag{36}$$

Analogously to (25), we assume that $p_{q,q}^{r(n)}$ is of affine structure:

$$p_{q,q}^{r(n)} = -D_n^r - H_n^{r\prime} X_q. \tag{37}$$

For $n = 0$ we have that:

$$p_{q,q}^{r(0)} = \ln P_{q,q}^{r(0)} = \ln 1 = 0, \tag{38}$$

which by equation (37) always holds if

$$\begin{aligned} D_0^r &= 0, \\ H_0^r &= 0. \end{aligned} \tag{39}$$

We will assume that (37) is valid for n and deduce its validity for $n + 1$.

According to the general pricing formula (21), and using (34):

$$P_{q,q}^{r(n+1)} = \mathrm{E}_q \left[P_{q,q+1}^{r(n)} \exp m_{q+1} \right] = \mathrm{E}_q \left[\exp(\pi_{q+1} + p_{q+1,q+1}^{r(n)} + m_{q+1}) \right]. \tag{40}$$

Applying (37) we get:

$$\begin{aligned} p_{q,q}^{r(n+1)} &= \mathrm{E}_q \left[\pi_{q+1} + p_{q+1,q+1}^{r(n)} + m_{q+1} \right] + 1/2 \operatorname{Var}_q \left[\pi_{q+1} + p_{q+1,q+1}^{r(n)} + m_{q+1} \right] \\ &= \mathrm{E}_q \left[\pi_{q+1} \right] \\ &\quad - D_n^r - H_n^{r\prime} \mathrm{E}_q \left[X_{q+1} \right] \\ &\quad + \mathrm{E}_q \left[m_{q+1} \right] \\ &\quad + \frac{1}{2} Var_q \left[\pi_{q+1} - H_n^{r\prime} X_{q+1} \right] \\ &\quad + \frac{1}{2} \operatorname{Var}_q \left[m_{q+1} \right] \\ &\quad + \operatorname{Cov}_q \left[\pi_{q+1} - H_n^{r\prime} X_{q+1}, m_{q+1} \right]. \end{aligned} \tag{41}$$

Let e_π indicate the position of the inflation in the state vector:

$$e_\pi = (0, 0, 1, 0)', \tag{42}$$

so that

$$\pi_{q+1} = e_\pi' X_{q+1}. \tag{43}$$

Using (43), (23), (24) and (11) we can rewrite (41) as

$$
\begin{aligned}
p_{q,q}^{r(n+1)} &= \mathrm{E}_q\left[\pi_{q+1} + p_{q+1,q+1}^{r(n)} + m_{q+1}\right] + 1/2\,\mathrm{Var}_q\left[\pi_{q+1} + p_{q+1,q+1}^{r(n)} + m_{q+1}\right] \\
&= e_\pi'\left((I-\Gamma)\mu + \Gamma X_q\right) \\
&\quad - D_n^r - H_n^{r\prime}\left((I-\Gamma)\mu + \Gamma X_q\right) \\
&\quad - e_y' X_q - \tfrac{1}{2}(\beta_0 + \beta_1 X_q)'\Sigma(\beta_0 + \beta_1 X_q) \\
&\quad + \frac{1}{2}(e_\pi' - H_n^{r\prime})\Sigma(e_\pi - H_n^r) \\
&\quad + \frac{1}{2}(\beta_0 + \beta_1 X_q)'\Sigma(\beta_0 + \beta_1 X_q) \\
&\quad + (H_n^{r\prime} - e_\pi')\Sigma(\beta_0 + \beta_1 X_q).
\end{aligned}
\tag{44}
$$

where the last part follows from

$$
\begin{aligned}
\mathrm{Cov}_q\left[\pi_{q+1} - H_n^{r\prime}X_{q+1}, m_{q+1}\right] &= \mathrm{Cov}_q\left[(e_\pi' - H_n^{r\prime})X_{q+1}, m_{q+1}\right] \\
&= \mathrm{Cov}_q[(e_\pi' - H_n^{r\prime})\left((I-\Gamma)\mu + \Gamma X_q + \varepsilon_{q+1}\right), \\
&\quad - e_y' X_q - \frac{1}{2}(\beta_0 + \beta_1 X_q)'\Sigma(\beta_0 + \beta_1 X_q) - (\beta_0 + \beta_1 X_q)'\varepsilon_{q+1}] \\
&= \mathrm{E}_q\left[((e_\pi' - H_n^{r\prime})\varepsilon_{q+1})(-(\beta_0 + \beta_1 X_q)'\varepsilon_{q+1})\right] \\
&= \mathrm{E}_q\left[((H_n^{r\prime} - e_\pi')\varepsilon_{q+1})(\varepsilon_{q+1}'(\beta_0 + \beta_1 X_q))\right] \\
&= (H_n^{r\prime} - e_\pi')\,\mathrm{E}_q\left[\varepsilon_{q+1}\varepsilon_{q+1}'\right](\beta_0 + \beta_1 X_q) \\
&= (H_n^{r\prime} - e_\pi')\Sigma(\beta_0 + \beta_1 X_q).
\end{aligned}
\tag{45}
$$

We can further rewrite $p_{q,q}^{r(n+1)}$ as

$$
\begin{aligned}
p_{q,q}^{r(n+1)} &= -D_n^r + (e_\pi' - H_n^{r\prime})(I-\Gamma)\mu + (e_\pi' - H_n^{r\prime})\Gamma X_q \\
&\quad - e_y' X_q \\
&\quad + \frac{1}{2}(e_\pi - H_n^r)'\Sigma(e_\pi - H_n^r) \\
&\quad + (H_n^{r\prime} - e_\pi')\Sigma\beta_0 + (H_n^{r\prime} - e_\pi')\Sigma\beta_1 X_q \\
&= -D_n^r + (e_\pi' - H_n^{r\prime})(I-\Gamma)\mu + \frac{1}{2}(e_\pi - H_n^r)'\Sigma(e_\pi - H_n^r) + (H_n^{r\prime} - e_\pi')\Sigma\beta_0 \\
&\quad + ((e_\pi' - H_n^{r\prime})\Gamma - e_y' + (H_n^{r\prime} - e_\pi')\Sigma\beta_1)X_q.
\end{aligned}
\tag{46}
$$

The last equation is already of the affine structure as in equation (25) with parameters

$$
\begin{aligned}
D_{n+1}^r &= D_n^r + (H_n^r - e_\pi)'(I-\Gamma)\mu - \frac{1}{2}(H_n^r - e_\pi)'\Sigma(H_n^r - e_\pi) - (H_n^r - e_\pi)'\Sigma\beta_0, \\
H_{n+1}^r &= e_y + (\Gamma - \Sigma\beta_1)'(H_n^r - e_\pi).
\end{aligned}
\tag{47}
$$

B.1.3 Stocks

The excess return on stocks is defined as follows and can be rearranged using (32):

$$r_{q+1} - y_q^{(1)} = \ln\left(\frac{P_{q+1}}{P_q}\right) - y_q^{(1)} = p_{q+1} - p_q + p_q^{(1)} \tag{48}$$

$$\implies p_{q+1} = r_{q+1} - y_q^{(1)} + p_q - p_q^{(1)}, \tag{49}$$

where P_q is the stock price and r_q is its return. Note that both r_{q+1} and $y_q^{(1)}$ are returns going from period q to period $q+1$, with $y_q^{(1)}$ being known in period q. Using (30) in (28) we get

$$p_q^{(1)} = \mathrm{E}_q\left[m_{q+1}\right] + \frac{1}{2}\mathrm{Var}_q\left[m_{q+1}\right]. \tag{50}$$

From (22), (48) and (50) it follows that

$$
\begin{aligned}
\mathrm{E}_q\left[r_{q+1} - y_q^{(1)}\right] &= \mathrm{E}_q[p_{q+1}] - \left(\mathrm{E}_q\left[p_{q+1} + m_{q+1}\right] + 1/2\,\mathrm{Var}_q\left[p_{q+1} + m_{q+1}\right]\right) \\
&\quad + \mathrm{E}_q\left[m_{q+1}\right] + \frac{1}{2}\mathrm{Var}_q\left[m_{q+1}\right] \\
&= \mathrm{E}_q[p_{q+1}] - \left(\mathrm{E}_q\left[p_{q+1}\right] + \frac{1}{2}\mathrm{Var}_q\left[p_{q+1}\right] + \mathrm{Cov}_q\left[p_{q+1}, m_{q+1}\right]\right) \\
&= -\frac{1}{2}\mathrm{Var}_q\left[p_{q+1}\right] - \mathrm{Cov}_q\left[p_{q+1}, m_{q+1}\right].
\end{aligned}
$$

Using (49) and the fact that p_q and $p_q^{(1)}$ are known at time q, so that

$$\mathrm{Var}_q[p_{q+1}] = \mathrm{Var}_q[r_{q+1} - y_q^{(1)} + p_q - p_q^{(1)}] = \mathrm{Var}_q[r_{q+1} - y_q^{(1)}], \tag{51}$$

we get

$$\mathrm{E}_q\left[r_{q+1} - y_q^{(1)}\right] = -\frac{1}{2}\mathrm{Var}_q\left[r_{q+1} - y_q^{(1)}\right] - \mathrm{Cov}_q\left[r_{q+1} - y_q^{(1)}, m_{q+1}\right]. \tag{52}$$

The excess return on stocks can also be written as

$$r_{q+1} - y_q^{(1)} = e_{xs}' X_{q+1} = e_{xs}'\left((\mathrm{I} - \Gamma)\mu + \Gamma X_q + \varepsilon_{q+1}\right), \tag{53}$$

where $e_{xs} = (0,1,0,0)'$ is a unit vector representing the location of the excess return on stocks in the state vector. Hence,

$$e_{xs}'\left((\mathrm{I} - \Gamma)\mu + \Gamma X_q\right) = \mathrm{E}_q\left[r_{q+1} - y_q^{(1)}\right]. \tag{54}$$

It follows from equation (20), (52), (53) and (54) that

$$
\begin{aligned}
e'_{xs}\left((I-\Gamma)\mu + \Gamma X_q\right) &= -\frac{1}{2}\operatorname{Var}_q\left[e'_{xs}\varepsilon_{q+1}\right] - \operatorname{Cov}_q\left[e'_{xs}\varepsilon_{q+1}, -(\beta_0 + \beta_1 X_q)'\varepsilon_{q+1}\right] \\
&= -(\frac{1}{2}e'_{xs}\Sigma e_{xs}) - \operatorname{E}_q\left[(e'_{xs}\varepsilon_{q+1} - 0)(-(\beta_0 + \beta_1 X_q)'\varepsilon_{q+1} - 0)\right] \\
&= -(\frac{1}{2}e'_{xs}\Sigma e_{xs}) - \operatorname{E}_q\left[-e'_{xs}\varepsilon_{q+1}\varepsilon'_{q+1}(\beta_0 + \beta_1 X_q)\right] \\
&= -(\frac{1}{2}e'_{xs}\Sigma e_{xs}) + e'_{xs}\Sigma(\beta_0 + \beta_1 X_q).
\end{aligned}
$$

Hence,

$$
e'_{xs}\left((I-\Gamma)\mu + \Gamma X_q + \frac{1}{2}\Sigma e_{xs} - \Sigma\beta_0 - \Sigma\beta_1 X_q\right) = 0
$$

$$
\Leftrightarrow e'_{xs}\left(\left((I-\Gamma)\mu - \Sigma\beta_0 + \frac{1}{2}\Sigma e_{xs}\right) + (\Gamma - \Sigma\beta_1)X_q\right) = 0.
$$

This is satisfied for all values of X_q if:

$$
\begin{aligned}
e'_{xs}\left((I-\Gamma)\mu - \Sigma\beta_0\right) + \frac{1}{2}e'_{xs}\Sigma e_{xs} &= 0, \\
e'_{xs}\left(\Gamma - \Sigma\beta_1\right) &= 0.
\end{aligned}
\tag{55}
$$

The first equation yields one condition, whereas the second equation yields as many conditions as there are state variables. It follows that the conditions in (55) must be satisfied for stocks. These conditions determine the parameters β_0 and β_1 of the discount factor.

B.1.4 Inflation risk premium

The inflation risk premium $RP_q^{(n)}$ is defined by the following equation:

$$
y_q^{(n)} - y_{q,q}^{r(n)} = \frac{1}{n}\sum_{i=1}^{n}\operatorname{E}_q[\pi_{q+i}] + \frac{1}{2}e'_\pi\Sigma e_\pi + RP_q^{(n)},
\tag{56}
$$

where on the left hand side we have the difference between the nominal and the real yield, and on the right hand side the first term is n-period expected inflation and the second term is the so-called Jensen's correction - a convexity term that has little impact on the results.

Using the formula of the sum of geometric series we can express X_{q+i}

$$
\begin{aligned}
\operatorname{E}_q[X_{q+i}] &= \Gamma^0(I-\Gamma)\mu + \Gamma^1(I-\Gamma)\mu + ... + \Gamma^{i-1}(I-\Gamma)\mu + \Gamma^i X_q \\
&= (I-\Gamma)\mu(I-\Gamma^i)(I-\Gamma)^{-1} + \Gamma^i X_q \\
&= \mu + \Gamma^i(X_q - \mu).
\end{aligned}
\tag{57}
$$

It follows then that the expected inflation is

$$E_q[\pi_{q+i}] = E_q[e'_\pi X_{q+i}] = e'_\pi \mu + e'^i_\pi(X_q - \mu), \tag{58}$$

and

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^n E_q[\pi_{q+i}] &= \frac{1}{n}\sum_{i=1}^n e'_\pi \mu + \frac{1}{n}\sum_{i=1}^n e'_\pi \Gamma^i(X_q - \mu) \\
&= e'_\pi \mu + \frac{1}{n}e'_\pi(\sum_{i=1}^n \Gamma^i(X_q - \mu)) \\
&= e'_\pi \mu + \frac{1}{n}e'_\pi((X_q - \mu)(I - \Gamma^{n+1})(I - \Gamma)^{-1} - (X_q - \mu)) \\
&= e'_\pi \mu + \frac{1}{n}e'_\pi \Gamma(I - \Gamma^n)(I - \Gamma)^{-1}(X_q - \mu).
\end{aligned} \tag{59}$$

Because

$$y_q^{(n)} = \frac{1}{n}D_n + \frac{1}{n}H'_n X_q, \tag{60}$$

and

$$y_{q,q}^{r(n)} = \frac{1}{n}D_n^r + \frac{1}{n}H_n^{r'}X_q, \tag{61}$$

inserting it into (56), using (59) and multiplying by n we get:

$$D_n + H'_n X_q - D_n^r - H_n^{r'}X_q = ne'_\pi \mu + e'_\pi \Gamma(I - \Gamma^n)(I - \Gamma)^{-1}(X_q - \mu) + \frac{n}{2}e'_\pi \Sigma e_\pi + nRP_q^{(n)}. \tag{62}$$

$$\begin{aligned}
0 &= D_n^r - D_n + ne'_\pi \mu - e'_\pi \Gamma(I - \Gamma^n)(I - \Gamma)^{-1}\mu + \frac{n}{2}e'_\pi \Sigma e_\pi + nRP_q^{(n)} \\
&\quad + (H_n^{r'} - H'_n + e'_\pi \Gamma(I - \Gamma^n)(I - \Gamma)^{-1})X_q.
\end{aligned} \tag{63}$$

If we assume that the risk premium is constant and independent of maturity we have that

$$nRP = (D_n - D_n^r - ne'_\pi \mu + e'_\pi \Gamma(I - \Gamma^n)(I - \Gamma)^{-1}\mu - \frac{n}{2}e'_\pi \Sigma e_\pi) + (H'_n - H_n^{r'} - e'_\pi \Gamma(I - \Gamma^n)(I - \Gamma)^{-1})X_q, \tag{64}$$

for all X_q. Hence,

$$\begin{aligned}
nRP &= (D_n - D_n^r - ne'_\pi \mu + e'_\pi \Gamma(I - \Gamma^n)(I - \Gamma)^{-1}\mu - \frac{n}{2}e'_\pi \Sigma e_\pi), \\
&\quad (H'_n - H_n^{r'} - e'_\pi \Gamma(I - \Gamma^n)(I - \Gamma)^{-1}) = 0.
\end{aligned} \tag{65}$$

Using (39) and (47)

$$\begin{aligned}
D_1^r &= -e'_\pi(I - \Gamma)\mu - \frac{1}{2}e'_\pi \Sigma e_\pi + e'_\pi \Sigma \beta_0, \\
H_1^r &= e_y - (\Gamma - \Sigma \beta_1)'e_\pi.
\end{aligned} \tag{66}$$

we can write for $n = 1$:

$$RP = D_1 - D_1^r - e_\pi'\mu + e_\pi'\Gamma\mu - \frac{1}{2}e_\pi'\Sigma e_\pi,$$
$$H_1^{r\prime} = H_1' - e_\pi'\Gamma. \tag{67}$$

Using (31) and (66)

$$RP = e_\pi'(I - \Gamma)\mu + \frac{1}{2}e_\pi'\Sigma e_\pi - e_\pi'\Sigma\beta_0 - e_\pi'\mu + e_\pi'\Gamma\mu - \frac{1}{2}e_\pi'\Sigma e_\pi,$$
$$e_y - (\Gamma - \Sigma\beta_1)'e_\pi = e_y - \Gamma'e_\pi. \tag{68}$$

$$e_\pi'(\Sigma\beta_0) = -RP,$$
$$e_\pi'(\Sigma\beta_1) = 0. \tag{69}$$

Since estimates of inflation risk premium vary from negative to positive (Grishchenko and Huang, 2012), we set RP to minus 20 basis points that follow from our data.

We also assume that the real wage growth risk premium is equal to zero. By following analogous procedure as for inflation risk premium we would obtain the following constraints:

$$e_w'(\Sigma\beta_0) = 0,$$
$$e_w'(\Sigma\beta_1) = 0, \tag{70}$$

where
$$e_w = (0, 0, 0, 1)'. \tag{71}$$

B.2   Parameter optimization

The empirical values $\hat{\alpha}$ and $\hat{\Gamma}$ for α and Γ in equation (8) are determined by regressing the state variables on their lagged values, using multivariate OLS. Further, $\hat{\Sigma}$ is the variance of the residuals, i.e. the difference between the observed and the fitted values. We use these values of $\hat{\alpha}$, $\hat{\Gamma}$ and $\hat{\Sigma}$ in the sequel. Given $\hat{\Gamma}$ and $\hat{\alpha}$, the estimate $\hat{\mu}$ of the average μ can be obtained from (10):

$$\hat{\mu} = (I - \hat{\Gamma})^{-1}\hat{\alpha}. \tag{72}$$

The procedure for obtaining the model-induced values $\tilde{\beta}_0$ and $\tilde{\beta}_1$ for β_0, respectively β_1, to determine the stochastic discount factor is more complicated. First, we obtain empirical estimates $\hat{D}_n$ and $\hat{H}_n$ of D_n and H_n, respectively, through multivariate OLS estimation of equation (27) for some specific maturities using the historical time series of zero-coupon yields for those maturities and using the historical time series of the state variables. The estimation is again at the quarterly level from the third quarter of 1971 up to and including the last quarter of 2012.

The same procedure is repeated for the real yields, in order to obtain empirical estimates $\hat{D}_n^r$ and $\hat{H}_n^r$. For this purpose we use the historic data of the real yields starting from 2003. The optimal $\tilde{\beta}_0$ and $\tilde{\beta}_1$ are obtained through an optimization procedure exploiting several model-implied restrictions. The optimal values for $\tilde{\beta}_0$ and $\tilde{\beta}_1$ imply specific values $\tilde{D}_n$, $\tilde{H}_n$, $\tilde{D}_n^r$ and $\tilde{H}_n^r$ for the model parameters D_n, H_n, D_n^r and H_n^r, respectively, by using the recursion in equations (29) and (47). Specifically, given $H_0 = 0$ and $\tilde{\beta}_1$, we can calculate $\tilde{H}_1$, $\tilde{H}_2$, and so on. Given $D_0 = 0$, $\tilde{\beta}_0$ and the path H_0, $\tilde{H}_1$, $\tilde{H}_2$,..., we calculate $\tilde{D}_1$, $\tilde{D}_2$, ... Similarly, given $H_0^r = 0$ and $\tilde{\beta}_1$, we can calculate $\tilde{H}_1^r$, $\tilde{H}_2^r$, and so on. Given $D_0^r = 0$, $\tilde{\beta}_0$ and the path H_0^r, $\tilde{H}_1^r$, $\tilde{H}_2^r$,..., we calculate $\tilde{D}_1$, $\tilde{D}_2$, ... The objective function constructed in order to solve for $\tilde{\beta}_1$ aims at matching the historic yield exposures to the state variables to the exposures in the model, i.e. bringing $\tilde{H}_n$ and $\tilde{H}_n^r$ "as close as possible" to $\hat{H}_n$ and $\hat{H}_n^r$. The objective function constructed in order to solve for $\tilde{\beta}_0$ mainly aims at matching the last observation of interest rate levels in reality to the ones in the model, i.e. bringing $\tilde{D}_n$ and $\tilde{D}_n^r$ as close as possible to $\hat{D}_n$ and $\hat{D}_n^r$. Once we have the series $\tilde{D}_n$ and $\tilde{H}_n$, we have constructed the nominal term structure that we use to calculate the returns on the fixed-income part of pension fund portfolio.

B.2.1 Optimization over β_1

Note that the model constraints in the second lines of (55), (69) and (70) are expressed in terms of $\Sigma\beta_1$, so it is easier to estimate the combination $\Sigma\beta_1$ than β_1 itself.

The constraints on $\Sigma\beta_1$ are satisfied analytically by setting the values in the second row (corresponding to excess stock returns) of $\Sigma\beta_1$ to the corresponding values in the $\hat{\Gamma}$ matrix (constraint (55)) and by setting the third and the fourth rows (corresponding to inflation and real wage growth, respectively) of $\Sigma\beta_1$ to zeros (constraints (69) and (70)). These three constraints are imposed before minimising the objective function below over the remaining parameters in $\Sigma\beta_1$.

Then, we obtain $\tilde{\beta}_1$ in two steps. The first step is to obtain the optimal $\Sigma\beta_1$ that minimizes the criterion function, which is a sum of two components.

The first component is meant to match the nominal yield exposures to state variables as they follow from the model to their empirical values:

$$\sum_{n\in\tau} \left\| H_n - \hat{H}_n \right\|^2 = \sum_{n\in\tau} \left\| e_y + (\hat{\Gamma} - \Sigma\beta_1)' H_{n-1} - \hat{H}_n \right\|^2, \tag{73}$$

where $\tau = \{4, 8, 20, 40\}$. Note that H_{n-1} is a function of $\Sigma\beta_1$ through the recursion in the second line of (29).

The second component is constructed analogously for the real yields:

$$\sum_{n\in\tau} \left\| H_n^r - \hat{H}_n^r \right\|^2 = \sum_{n\in\tau} \left\| e_y + (\hat{\Gamma} - \Sigma\beta_1)_{n-1}'^r - e_\pi) - \hat{H}_n^r \right\|^2, \tag{74}$$

where $\tau = \{4, 8, 20, 40\}$, where H_{n-1}^r is a function of $\Sigma\beta_1$ through the recursion in the second line of (47). The two components are equally weighted.

The second step is to calculate $\tilde{\beta}_1$ using the estimated $\hat{\Sigma}$ and the $(\Sigma\tilde{\beta}_1)$ that minimizes the above objective:

$$\tilde{\beta}_1 = \hat{\Sigma}^{-1}(\Sigma\tilde{\beta}_1). \tag{75}$$

Using $H_0 = 0$, $H_0^r = 0$ and the optimal value $\tilde{\beta}_1$ in combination with the second lines of (29) and (47) we can thus calculate $\tilde{H}_1$, $\tilde{H}_2$,... and $\tilde{H}_1^r$, $\tilde{H}_2^r$,..., which we will use in the remaining optimization procedure and further in the model.

B.2.2 Optimization over β_0

Now that we have solved for $\tilde{\beta}_1$, $\tilde{H}_n$ and $\tilde{H}_n^r$ we can try to find the value of $\tilde{\beta}_0$ for β_0 that brings the nominal and the real yields that follow from the model as close as possible to the historical empirical yields.

Note that, analogously to β_1, the model constraints in the first lines of (55), (69) and (70) are expressed in terms of $\Sigma\beta_0$, so it is easier to estimate the combination $\Sigma\beta_0$ than β_0 itself.

The first line of (55) implies that the $\Sigma\beta_0$ element corresponding to the excess stock returns has a fixed value:

$$e_{xs}'(\Sigma\beta_0) = e_{xs}'(\mathrm{I} - \hat{\Gamma})\hat{\mu} + \frac{1}{2}e_{xs}'\hat{\Sigma}e_{xs}. \tag{76}$$

Following the first lines of (69) and (70) we set the elements of $\Sigma\beta_0$ corresponding to price inflation and real wage growth to the assumed risk premium levels.

We set up the following objective function that is to be minimized over the remaining element of $\Sigma\beta_0$, and that consists of two components. We will make use of the affine structure of the yields:

$$y_q^{(n)} = \frac{1}{n}D_n + \frac{1}{n}H_n'X_q, \tag{77}$$

and

$$y_q^{r(n)} = \frac{1}{n}D_n^r + \frac{1}{n}H_n^{r\prime}X_q. \tag{78}$$

The first component of the objective function is meant to match the intercept D_n in the affine term structure as follows from the model to its empirical value:

$$\sum_{n \in \tau} \left\| D_n - \hat{D}_n \right\|^2 =$$
$$\sum_{n \in \tau} \left\| \left(D_{n-1} + \tilde{H}_{n-1}'(I - \hat{\Gamma})\hat{\mu} - \frac{1}{2}\tilde{H}_{n-1}'\hat{\Sigma}\tilde{H}_{n-1} - \tilde{H}_{n-1}'(\Sigma\beta_0) \right) - \hat{D}_n \right\|^2, \tag{79}$$

where $\tau = \{4, 8, 20, 40\}$ and we have used the first line of (29).

The second component is constructed analogously for the real yields:

$$
\sum_{n \in \tau} \left\| D_n^r - \hat{D}_n^r \right\|^2 =
$$

$$
\sum_{n \in \tau} \left\| \left(D_{n-1}^r + (\tilde{H}_{n-1}^r - e_\pi)'(I - \hat{\Gamma})\hat{\mu} - \frac{1}{2}(\tilde{H}_{n-1}^r - e_\pi)'\hat{\Sigma}(\tilde{H}_{n-1}^r - e_\pi) - (\tilde{H}_{n-1}^r - e_\pi)(\Sigma\beta_0) \right) - \hat{D}_n^r \right\|^2,
$$

(80)

where $\tau = \{4, 8, 20, 40\}$ and we have used the first line of (47).

After the optimal value $\Sigma\tilde{\beta}_0$ has been found, $\tilde{\beta}_0$ can be calculated as follows:

$$
\tilde{\beta}_0 = \hat{\Sigma}^{-1}(\Sigma\tilde{\beta}_0).
$$

(81)

Using the optimal value of $\tilde{\beta}_0$ and $D_0 = D_0^r = 0$ the sequences $\tilde{H}_n$ and $\tilde{H}_n^r$ we can thus construct $\tilde{D}_n$ and $\tilde{D}_n^r$ using the first lines in (29) and (47). We now have the term structure fully constructed.

B.3 Returns on fixed-income holdings

The model generates scenarios for the term structure of interest rates. The fixed-income portfolio returns have to be extracted from this information. We assume that the portfolio consists of fixed-coupon bonds with a principal value of one unit of currency to be repaid at maturity, τ years from now.

$$
1 = cp \sum_{t=1}^{\tau} \left(1 + Y_0^{(4t)} \right)^{-4t} + \left(1 + Y_0^{(4\tau)} \right)^{-4\tau},
$$

where cp is the annual coupon payment and we have assumed that the bond is priced at par. The interest rate $Y_0^{(4t)}$ is the quarterly interest rate obtained from the construction of the term structure using the above affine structure model. Hence, the left-hand side is the bond's price at issuance date when it is sold at par value 1, while the right-hand side is the present discounted value of the cash flows associated with the bond. These consist of the sum of the coupon payments discounted back to issuance date and the repayment of the principal, also discounted back to issuance date. Hence,

$$
cp = \frac{1 - \left(1 + Y_0^{(4\tau)} \right)^{-4\tau}}{\sum_{t=1}^{\tau} \left(1 + Y_0^{(4t)} \right)^{-4t}}.
$$

When the coupon is determined, the cash flows of the bond are known. Hence, by discounting them against the interest rate term structure at the beginning of the quarter and the term structure at the end of the quarter, the return can be calculated by subtracting the former

value from the latter value. Hence, the bond return is obtained as follows:

$$\left(cp\sum_{t=1}^{\tau}\left(1+Y_1^{(4t-1)}\right)^{-(4t-1)}+\left(1+Y_1^{(4\tau-1)}\right)^{-(4\tau-1)}\right)-\left(cp\sum_{t=1}^{\tau}\left(1+Y_0^{(4t)}\right)^{-4t}+\left(1+Y_0^{(4\tau)}\right)^{-4\tau}\right),$$

divided by the purchase price of the bond, which is one.

For the fixed-income portfolio we assume a portfolio consisting of 10-year maturity bonds that are priced at par. Hence, $\tau = 10$. The portfolio is rebalanced at the beginning of every time period so that it again consists of 10-year maturity bonds.

C   Risk-neutral sampling

The price of a derivative paying a cash flow Z (which is a function of the path $X_1, X_2, ... X_\tau$ of the state vector) at time τ is

$$P_0 = \mathrm{E}_0\left[Z_\tau \exp\sum_{q=1}^{\tau} m_q\right].$$

Using (20) this equation can be rewritten as

$$P_0 = \int Z(\varepsilon_1, \varepsilon_2, ..., \varepsilon_\tau)\exp\left[-\sum_{q=1}^{\tau} e_y' X_{q-1}\right]$$

$$\exp\left[-\sum_{q=1}^{\tau}\left(\frac{1}{2}(\beta_0+\beta_1 X_{q-1})'\Sigma(\beta_0+\beta_1 X_{q-1})+(\beta_0+\beta_1 X_{q-1})'\varepsilon_q\right)\right]$$

$$\frac{1}{(2\pi)^{k\tau/2}|\Sigma|^{\tau/2}}\exp\left[-\sum_{q=1}^{\tau}\frac{1}{2}\varepsilon_q'\Sigma^{-1}\varepsilon_q\right]d\varepsilon_1 d\varepsilon_2...d\varepsilon_\tau.$$

where k is the dimension of the state vector (in our case, 4). Hence,

$$P_0 = \int Z(\varepsilon_1, \varepsilon_2, ..., \varepsilon_\tau)\exp\left[-\sum_{q=1}^{\tau} y_{q-1}^{(1)}\right]\frac{1}{(2\pi)^{k\tau/2}|\Sigma|^{\tau/2}}$$

$$\exp\left[-\sum_{q=1}^{\tau}\left(\frac{1}{2}\left(\varepsilon_q+\Sigma(\beta_0+\beta_1 X_{q-1})\right)'\Sigma^{-1}\left(\varepsilon_q+\Sigma(\beta_0+\beta_1 X_{q-1})\right)\right)\right]d\varepsilon_1 d\varepsilon_2...d\varepsilon_\tau.$$

This integral can be evaluated numerically in a Monte Carlo simulation by drawing a number of time series $\{\varepsilon_1, \varepsilon_2, ..., \varepsilon_\tau\}$ from the multivariate normal density function

$$f = \frac{1}{(2\pi)^{k\tau/2}|\Sigma|^{\tau/2}}\exp\left[-\sum_{q=1}^{\tau}\left(\frac{1}{2}\left(\varepsilon_q+\Sigma(\beta_0+\beta_1 X_{q-1})\right)'^{-1}\left(\varepsilon_q+\Sigma(\beta_0+\beta_1 X_{q-1})\right)\right)\right], \quad (82)$$

calculating for each time series the derivative payoff discounted at the risk-free rate

$$\exp\left[-\sum_{q=1}^{\tau} y_{q-1}^{(1)}\right] \cdot Z(\tilde{\varepsilon}_1, \tilde{\varepsilon}_2, ..., \tilde{\varepsilon}_\tau),$$

and taking a simple average over the drawings. Hence, we have transformed the original problem in which we draw from the multivariate normal density function of ε_q and discount cash flows at the stochastic discount factor into a problem in which we draw from the multivariate normal density function of $\tilde{\varepsilon}_q$ with mean $-\Sigma(\beta_0 + \beta_1 X_{q-1})$ and the same original variance-covariance matrix Σ, but discount cash flows at the risk-free rate.

We can draw $\tilde{\varepsilon}_1, \tilde{\varepsilon}_2, ..., \tilde{\varepsilon}_\tau$ from the distribution $f(\tilde{\varepsilon}_1, \tilde{\varepsilon}_2, ..., \tilde{\varepsilon}_\tau)$ by first drawing $\tilde{\varepsilon}_1$ from its marginal distribution, then drawing $\tilde{\varepsilon}_2$ from the conditional distribution $f(\tilde{\varepsilon}_2|\tilde{\varepsilon}_1)$, and so on. The conditional density $f(\tilde{\varepsilon}_q|\tilde{\varepsilon}_{q-1}, ..., \tilde{\varepsilon}_1)$ is obtained using the Bayes' formula:

$$f(\tilde{\varepsilon}_q|\tilde{\varepsilon}_{q-1}, ..., \tilde{\varepsilon}_1) = \frac{f(\tilde{\varepsilon}_q, ..., \tilde{\varepsilon}_1)}{f(\tilde{\varepsilon}_{q-1}, ..., \tilde{\varepsilon}_1)}.$$

Applying (82) to $\tau = q - 1$ and $\tau = q$, we obtain

$$f(\tilde{\varepsilon}_q|\tilde{\varepsilon}_{q-1}, ..., \tilde{\varepsilon}_1) =$$
$$\frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}} \exp\left[-\frac{1}{2}\left(\tilde{\varepsilon}_q + \Sigma(\beta_0 + \beta_1 X_{q-1})\right)' \Sigma^{-1} \left(\tilde{\varepsilon}_q + \Sigma(\beta_0 + \beta_1 X_{q-1})\right)\right].$$

Hence,

$$\tilde{\varepsilon}_q|\tilde{\varepsilon}_{q-1}, ..., \tilde{\varepsilon}_1 \sim \mathbb{N}(-\Sigma(\beta_0 + \beta_1 X_{q-1}), \Sigma).$$

D   Generating scenarios

The economic scenarios are generated using the parameters $\hat{\alpha}$, $\hat{\Gamma}$, $\hat{\Sigma}$, $\tilde{\beta}_0$, $\tilde{\beta}_1$, $\tilde{D}_n$, $\tilde{H}_n$, $\tilde{D}_n^r$ and $\tilde{H}_n^r$.

First, setting the initial value of the state vector at the average μ and using (8) the path of the vector of state variables is simulated for the chosen horizon length. Then, at each quarter into the horizon, using (27) the term structure of the nominal interest rate is constructed.

The same scenario-generating procedure is followed for both the real-world and risk-neutral scenarios. The only difference lies in the dynamics of shock ε. For the real-world simulation the error terms are drawn from the mean-zero normal distribution $\varepsilon_{q+1} \sim \mathbb{N}(0, \Sigma)$ given by (9). Under the risk-neutral scenarios, the noise term is normally distributed with $\tilde{\varepsilon}_{q+1} \sim \mathbb{N}(-\Sigma(\beta_0 + \beta_1 X_q), \Sigma)$.