Longevity-risk transfer with financial risk: is it worth for annuity providers?*

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May 8, 2013

Abstract

This paper formalizes the trade-off between the cost and benefits of longevity risk transfer for an annuity provider, taking into account both mortality and interest-rate risk appraisal, on both assets and liabilities. The main novelty of the paper consists in providing an overall Value-at-Risk (VaR) in closed form, using a simplified, consistent way to measure both interest-rate and longevity risk. A risk-return frontier built, using VaR shows that the transfer may increase or decrease the overall risk, while decreasing returns. This applies to the 2010 UK annuity and bond market. A utility-based decision criterion is used to locate the optimal transfer.

JEL Classification: G22, G32.

1 Introduction

Longevity risk - which is the risk of unexpected improvements in survivorship - is known to be an important threat to the safety of annuity providers, such as pension funds. These institutions run the risk of seeing their liabilities increase over time, when the actual survival rate of their members is greater than the forecasted one. As of 2007, the exposure of pension funds and other annuity providers to unexpected improvements in life expectancy has been quantified in 400 billion USD for the US and UK, more than 20 trillion USD worldwide (see Biffis and Blake (2010)).

*The Authors thank Antonella Basso, Michael Brennan, Giovanna Nicodano, Ermanno Pitacco, Richard Stapleton (discussant) and conference participants to the Cerp Conference 2012, Moncalieri, September 2012, the XV Workshop in Quantitative Finance, Rimini, January 2013, the PARTY 2013 Conference, Ascona, January 2013, the Afmath Conference 2013, Brussels, February 2013, the Marie Curie Conference, Konstanz, April 2013, for helpful comments.

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Annuity providers are also exposed to financial risks on both assets and liabilities, as soon as the latter are fairly evaluated. Consider interest-rate risk, which is the risk that realized interest rates differ from forecasted ones. This risk affects both the fraction of assets invested in bonds and the fair value of liabilities, since in the latter future cash-flows are discounted using market interest rates. As long as liabilities were not evaluated at fair value, there was no impact of market interest-rate risk on liabilities. Nowadays, the IASB (International Accounting Standard Board) forces evaluation of liabilities at fair value. Regulatory provisions of the Solvency II type require to align capital standards to the market value of liabilities. Also assets need to be evaluated at their market value. Given the current regulatory and accounting rules, then, it is important to evaluate the interest-rate risk effect on both assets and liabilities.

Regulatory and accounting interventions make an holistic view of longevity and financial risk matter, since liabilities are subject to both, even when assets are subject to financial risk only. Numerical approaches to the problem have been the object of many efforts, both in the industry and in the Academia. Successful simulation tools exist. At the opposite, modelling financial and longevity risk in closed form, while looking at both assets and liabilities, seems to be a very hard task. Standard longevity models indeed are usually cast in discrete-time and, as such, lack analytical tractability when combined with the standard, continuous-time models for interest-rate risk.

In order to obtain closed-form evaluations, we choose a parsimonious continuous-time model for longevity risk, together with a standard model for interest rates. The description of the actuarial and financial market allows us to obtain easy-to-compute analytical expressions for both the expected return and the risk associated to the portfolio of assets and liabilities of the annuity provider or pension fund. Within this holistic view, we ask ourselves whether it should better transfer longevity risk to a reinsurer or a special purpose vehicle - as most of the recent deals do - or remaining exposed to it, while saving on the costs of the transfer.

Given this approach, our paper stands in between modern demographic-risk transfer contributions and Asset-Liability Management (ALM) studies in Insurance. Indeed, a flourishing literature studies the possibility of transferring demographic risk through the financial market, using s-forwards, q-forwards or other over-the-counter products. This literature focuses on ways to alleviate the cost of the transfer, or make it fairly priced, and most of the time considers also basis risk. In order to do so, it tends to abstract away from financial risk and optimal retention. Those papers which do consider optimal retention tend to stress the role of systematic versus idiosyncratic risk, and to isolate it from financial risk. Two important and novel theoretical approaches in this sense are Biffis and Blake (2010) and Barrieu and Loubergé (2013). Biffis and Blake (2010) address the issue of optimal retention in equilibrium, by focussing on asymmetric information about longevity risk, in the presence of systematic and idiosyncratic risk. Since the focus is on information effects, optimal retention does not depend on its impact on financial risk exposure, as in our context, but only on its direct effects on capital requirements. Barrieu and Loubergé (2013) study optimal longevity risk transfers by explicitly modelling the alternative between reinsurance and securitization. Even in their case, a crucial role is played by regulatory constraints – which differentiate reinsurance from securitization - when both systematic and idiosyncratic longevity risk exist. Since the focus is on pure longevity risk sharing, they do not consider interactions between longevity risk transfer and financial risk. We depart from these two papers by starting from the consideration that liabilities, once fairly priced, suffer interest-rate risk exactly as assets, and we aim at capturing the financial risk impact of reinsurance.

At the same time, our paper differs from most of the literature in pension fund's asset-liability management, since we capture systematic longevity risk by modelling the intensity of mortality of annuitants as a stochastic process instead of a (known) constant. Most of the ALM literature neglects systematic mortality risk, by assuming that mortality intensity is deterministic, or presumes that it has been reinsured away. It assesses the impact of idiosyncratic risk, together with financial risk, on pension plan management decisions. This is the approach taken for instance by Hainaut and Devolder (2007) and Battocchio et al. (2007).

We neglect idiosyncratic longevity risk and focus on the worthiness of protecting against systematic changes in longevity, given the impact that reinsurance has on financial risk. In order to do that we build a simplified model of overall risk-return assessment, which allows for first-order approximations and provides closed-form expected return and Value-at-Risk for assets and liabilities. In our simplified context the fund's alternatives can be represented in the plane (VaR, Expected Return). The slope of the corresponding frontier is such that transfer may increase or decrease the overall risk, while reducing expected returns. The optimal transfer choices of the fund are located along the (efficient part) of the frontier. Once the preferences of the fund are described in terms of its risk-versus-return appetite, the optimal demand for reinsurance is determined using an expected-utility approach.

The paper is structured as follows. Section 2 formalizes our set up for both longevity and financial risk. Section 3 measures the effects of transferring versus retaining longevity risk, as well as the effects of financial risk. It obtains a minimum price for demographic risk transfer, based on the reinsurer's hedge and its fair price. Section 4 introduces a VaR measure for the overall risk of assets and liabilities and spells out the trade-offs between risk and return. Section 5 describes the different fund strategies in the plane (VaR, Expected Return) and the preferences of the fund. Section 6 provides a calibrated example using financial and demographic data from the UK market. Section 7 concludes and outlines extensions.

2 Set up

Let us place ourselves in a standard, continuous-time framework. Consider a time interval $\mathcal{T} = [0, T], T < \infty$, a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a multidimensional standard Wiener $W(\omega, t), t \in \mathcal{T}$. The space is endowed with the filtration generated by w, $\mathbf{F}^w = \{\mathcal{F}_t\}$. We adopt a stochastic extension of the classical Gompertz law for mortality description and we stick to the Hull-White model for interest-rate risk[1]. In doing so, we assume that the assets of the pension fund can be invested in risky bonds or kept as cash.

2.1 Demographic risk

Mortality risk - which, with a slight abuse of terminology, we call also longevity or demographic risk - exists since death occurs as a Poisson process, with an intensity which, instead of being deterministic as in the classical actuarial framework, is stochastic. This permits experienced mortality to be different from the forecasted one. At each point in time there is an actual mortality intensity, $\lambda(t)$, which may differ from its forecast at any previous point in time, the forward intensity. If the forecast is done at time 0, we denote it as $f(0,t)$. So, longevity risk arises because the actual intensity $\lambda(t)$ may differ from the forward mortality intensity $f(0,t)$, exactly as in the interest rate domain.

This stochastic intensity - or stochastic-mortality - approach, which is by-now quite well known in the literature, has the advantage of making closed-form evaluations, as well as description of age, period and cohort effects in mortality, possible. If the intensity is described by linear affine processes, the survival function is indeed known in closed form and can be calibrated using a parsimonious number of parameters. In order to stay in the linear class, to keep the distinction between age, period and cohort effects, but to be extremely parsimonious, we assume that the mortality intensity of a head aged x at calendar time t - which belongs to the generation born at time $i = t - x$ - is described by a so-called Ornstein-Uhlenbeck process, without mean reversion (OU):

$$d\lambda_i(t) = a_i \lambda_i(t) dt + \sigma_i dW_i(t),$$

where $a_i > 0$, $\sigma_i > 0$, W_i is a standard one-dimensional Brownian motion in W. In the notation we omit the dependence on x, since once calendar time and generation or gender are specified, age is uniquely determined.

This intensity extends - with the inclusion of a diffusive term - the classical Gompertz law

$$d\lambda_i(t) = a_i \lambda_i(t) dt,$$

where $a_i > 0$ is the rate of growth of the force of mortality. Expected intensity increases over age:

$$\mathbb{E}_t(\lambda_i(t + \Delta t)) = \lambda_i(t) \exp(a_i \Delta t) = f_i(t, t + \Delta t) + \frac{\sigma_i^2}{2a_i^2} \left[1 - \exp(a_i \Delta t)\right]^2, \quad (1)$$

The instantaneous volatility of death intensity is constant, while the overall variance increases exponentially in time:

$$\mathbb{V}ar_t(\lambda_i(t + \Delta t)) = -\frac{\sigma_i^2}{2a_i} \left[1 - \exp(2a_i \Delta t)\right]. \quad (2)$$

[1]This is the framework we adopted for measuring and hedging - but not for transferring - the Delta-Gamma risk of the reserves/liabilities in Luciano et al. (2012a).

With this construction, i.e. by assuming that there is an intensity process for each generation, we intend to capture longevity improvements or discrepancies over generations. We indeed have (and will calibrate) one drift and one diffusion for each generation (and gender, obviously). This - together with the OU choice - makes the overall mortality model flexible but still quite parsimonious. Empirical explorations have shown that it is however quite good in fitting actual per-cohort mortality (see Sherris and Wills (2008), among others).

Since it belongs to the affine class, the model provides a closed-form expression for the survival probability of generation i at any point in time t and up to any horizon T. Using a transformation from Jarrow and Turnbull (1994), the survival probability can be written as

$$\begin{aligned} S_i(t,T) &= \mathbb{E}_t\left[\exp\left(-\int_t^T \lambda_i(s)ds\right)\right] = \\ &= \frac{S_i(0,T)}{S_i(0,t)}\exp\left[-X_i(t,T)I_i(t) - Y_i(t,T)\right], \end{aligned}$$

where $S_i(0,T)$ and $S_i(0,t)$ are the survival frequencies at time 0,

$$X_i(t,T) := \frac{\exp(a_i(T-t)) - 1}{a_i},$$

$$Y_i(t,T) := -\frac{\sigma_i^2\left[1-\exp\left(2a_i t\right)\right]X_i(t,T)^2}{4a_i},$$

and $I_i(t)$ is the difference between the actual mortality intensity of generation i at t and its forward value or forecast at time $0, f_i(0,t)$. We interpret this difference as the *mortality or demographic risk factor*:

$$I_i(t) := \lambda_i(t) - f_i(0,t).$$

It is the discrepancy between realization and forecast which makes the pension fund exposed to mortality risk. Indeed, only the survival probabilities at the current date ($t = 0$) are known, while the probabilities which will be assigned at any future point in time ($t > 0$) are random variables. We will see below that this makes the reserves of the pension fund at any future point in time stochastic, and generates the demographic risk it has to cover. Randomness enters through the factor $I_i(t)$ and affects the whole survival curve, namely $S_i(t,T)$ for every T.

2.2 Financial risk

In order to seize the effects of interest rate changes on assets and liabilities we need to select a model for financial risk. The natural choice is to assume that interest rates follow an Ornstein-Uhlenbeck (OU) or constant-parameter Hull-and-White one-factor model. This is a standard choice in Financial modelling, able to provide us with closed form formulas for pricing and hedging, parsimonious but flexible enough to be popular in applications. In our context, it has the advantage of modelling financial risk in a fashion symmetric to demographic risk. Indeed, the instantaneous interest rate in the Hull-White model has the following dynamics under a measure $\mathbb{Q}$ equivalent to $\mathbb{P}$:

$$dr(t) = g(\theta - r(t))dt + \Sigma dW_F(t), \tag{3}$$

where $\theta, g > 0, \Sigma > 0$ and W_F is a univariate Brownian motion independent[2] of W_i for all i; θ is the long-run mean of the short-rate process, while the parameter g is the speed of mean reversion. As a consequence, the instantaneous rate has expectation and variance equal to

$$\mathbb{E}_t \left[r(t+\Delta t) \right] = r(t)e^{-g\Delta t} + \theta \left[1 - e^{-g\Delta t} \right], \tag{4}$$

$$\mathbb{V}ar_t(r(t+\Delta t)) = \frac{\Sigma^2}{2g} \left[1 - \exp\left(-2g\Delta t\right) \right]. \tag{5}$$

The corresponding zero-coupon bond price - if the bond is evaluated at t and has maturity T - is[3]

$$B(t,T) = \mathbb{E}_t^{\mathbb{Q}} \left[\exp\left(-\int_t^T r(s)ds \right) \right] =$$

$$= \frac{B(0,T)}{B(0,t)} \exp\left[-\bar{X}(t,T)K(t) - \bar{Y}(t,T) \right],$$

where $B(0,t), B(0,T)$ are the bond prices as observed at time 0 for durations t, T,

$$\bar{X}(t,T) := \frac{1 - \exp(-g(T-t))}{g},$$

$$\bar{Y}(t,T) := \frac{\Sigma^2}{4g} \left[1 - \exp(-2gt) \right] \bar{X}^2(t,T),$$

and the difference between the time-t actual and forward rate $R(0,t)$:

$$K(t) := r(t) - R(0,t)$$

is the *financial risk factor*, akin to the demographic factor $I_i(t)$. As in the longevity case, the financial risk factor is the difference between actual and forecasted rates for time t, where the forecast is done at time 0. It is the only source of randomness which affects bonds. It is clear that - for any maturity T - the bond value at any point in time $t > 0$ is random. Values at time $t = 0$ only are known.

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[2] Under the original measure we have $W = (W_1, W_2, ...W_N, W'_F)$ where N is the maximum number of generations alive in $\mathcal{T}$, while W'_F is the Brownian motion which corresponds to W_F according to Girsanov' theorem. No arbitrage and completeness is assumed to hold in the financial market. Independency of financial and actuarial risk is preserved because of the diffusive nature of uncertainty: see Luciano et al. (2012a) and Dhaene et al. (2013).

[3] The short-rate process is given directly under the risk-neutral measure, so that no assumption on the market price of financial risk is needed. The parameters of the interest-rate market will be calibrated accordingly.

3 Risk measurement

3.1 Demographic risk measurement

Consider an annuity issued on an individual of generation i, aged x at t. Make the annuity payment per period equal to one, for the sake of simplicity, and assume that the annuity is fairly priced and reserved. Since we assumed that financial and demographic risks (Brownian motions) are independent, with no risk premium for longevity risk[4] the cash flow of the annuity at tenor T has a fair value at time t equal to the product of the survival probability S and the discount factor B:

$$S_i(t,T)B(t,T)$$

The whole annuity - which lasts until the extreme age ω - is worth

$$V_i^A(t) = \sum_{u=t+1}^{\omega-x} S_i(t,u)B(t,u)$$

With no risk transfer, the fund would incur demographic risk, since at any point in time t the fair value and reserve $V_i^A(t)$ can change because the intensity process does. It can be shown that such change can be approximated up to the second order as follows:

$$\Delta V_i^{AM}(t) = \Delta_A^M(t)\Delta I_i(t) + \frac{1}{2}\Gamma_A^M(t)\Delta I_i^2(t), \tag{6}$$

where the Deltas and Gammas are

$$\Delta_A^M(t) = -\sum_{u=t+1}^{\omega-x} B(t,u)S_i(t,u)X_i(t,u) < 0,$$

$$\Gamma_A^M(t) = \sum_{u=t+1}^{\omega-x} B(t,u)S_i(t,u)[X_i(t,u)]^2 > 0.$$

Their signs show that the annuity value is decreasing and convex in the risk factor.

From now on, we take the point of view of a pension fund which issued such contract at a price $P \geq V_i^A(0)$ and can

- either run into demographic risk, evaluated at its first-order impact $\Delta_A^M(t)\Delta I_i(t)$, or
- transfer the risk to a reinsurer.

[4] Since there is no price for demographic risk, expectations of functionals of the intensity - such as the survival probability - under the historical measure $\mathbb{P}$ and the risk-neutral one/ones $\mathbb{Q}$ coincide. Extensions to constant risk premiums are trivial.

The price at which the transfer is done is determined as follows. We assume that - when risk is transferred to the reinsurer - the latter covers it using short death contracts in his portfolio, i.e. death contracts he issued or absorbed from insurers. This is the so-called natural hedging, which is likely to be feasible for reinsurers, given the diversification of their portfolios. We ask ourselves at what fair price the reinsurer can absorb the demographic risk of the annuity. To this end, we assume that coverage of risk is done by the reinsurer up to first-order changes. It Delta-covers risk[5] by using a position in $\mathbb{N}$ death contracts on individuals of the same generation, gender and age, as in Luciano et al. (2012b). At time t, a death contract which covers the period (t, T) is priced

$$V_i^D(t,T) = \sum_{u=t+1}^{T} B(t,u)\left[S(t,u-1) - S(t,u)\right],$$

and is affected by a change in mortality intensity ΔI_i as follows:

$$\Delta V_i^D(t,T) = \Delta_D^M(t,T)\Delta I_i(t) + \frac{1}{2}\Gamma_D^M(t,T)\Delta I_i^2(t),$$

where

$$\begin{aligned} \Delta_D^M(t,T) &= \sum_{u=t+1}^{T} B(t,u)\left[-S_i(t,u-1)X_i(t,u-1) + S_i(t,u)X_i(t,u)\right] > 0, \\ \Gamma_D^M(t,T) &= \sum_{u=t+1}^{T} B(t,u)\left[S_i(t,u-1)X_i^2(t,u-1) - S_i(t,u)X_i^2(t,u)\right] < 0. \end{aligned}$$

Their signs show that the death contract is increasing and concave in the risk factor. The position $\mathbb{N}$ is determined so that the Delta of the portfolio made by the annuity - in which the reinsurer is short, since he took demographic risk in charge - and the death contract is zero:[6]

$$(-\Delta_A^M + \mathbb{N}\Delta_D^M)\Delta I_i(t) = 0$$

$$\mathbb{N}(t,T) = \frac{\Delta_A^M}{\Delta_D^M} =$$

$$= -\frac{\sum_{u=t+1}^{\omega-x} B(t,u)S_i(t,u)X_i(t,u)}{\sum_{u=t+1}^{T} B(t,u)\left[-S_i(t,u-1)X_i(t,u-1) + S_i(t,u)X_i(t,u)\right]} < 0.$$

[5]We maintain the assumption of Delta - as opposite to Delta-Gamma - coverage for all risks below. In principle, going from delta to delta-gamma coverage just requires the use of additional death contracts and the introduction of more equations. No major conceptual difference seems to be at stake. For this reason, we disregard the extension in the whole paper.

[6]The reinsurer is short the death contract, since the annuity value increases when longevity is greater than forecasted, while the death value decreases. As a consequence, the increase in the payments to annuitants due to an unexpected shock in longevity is compensated by the decrease in the expected payments due to life-insurance policyholders.

The fair cost of such coverage is the value of the death contracts needed for hedging:

$$V(t,T) = -\mathbb{N}\sum_{u=t+1}^{T} B(t,u)\left[S_i(t,u-1) - S_i(t,u)\right] =$$

$$= \frac{\sum_{u=t+1}^{\omega-x} B(t,u)S_i(t,u)X_i(t,u)\sum_{u=t+1}^{T} B(t,u)\left[S_i(t,u-1) - S_i(t,u)\right]}{\sum_{u=t+1}^{T} B(t,u)\left[-S_i(t,u-1)X_i(t,u-1) + S_i(t,u)X_i(t,u)\right]}. \quad (7)$$

We assume that - in order to absorb demographic risk - the reinsurer charges the fund with a price C which is not smaller than the fair price, $C \geq V(t,T)$.

3.2 Financial risk measurement

The annuity value, which enters the liabilities, is subject to financial risk, since it is fairly priced. The effect of a change in K on the annuity is:

$$\Delta V_i^{AF}(t) = \Delta_A^F(t)\Delta K(t) + \frac{1}{2}\Gamma_A^F(t)\Delta K^2(t),$$

where

$$\Delta_A^F(t) = -\sum_{u=t+1}^{\omega-x} B(t,u)S_i(t,u)\bar{X}(t,u) < 0,$$

$$\Gamma_A^F(t) = \sum_{u=t+1}^{\omega-x} B(t,u)S(t,u)[\bar{X}(t,u)]^2 > 0.$$

The sign of these coefficients reveals that the annuity is decreasing and convex in discrepancies between the actual and forecasted interest rates, exactly as the bonds are.

Any bond which enters the assets of the fund - for the sake of simplicity we consider zero-coupon bonds only - is subject to financial risk. Its sensitivity to changes in K - ΔK - is well known:

$$\Delta_B^F(t,T) = -B(t,T)\bar{X}(t,T) < 0, \quad (8)$$

$$\Gamma_B^F(t,T) = B(t,T)\bar{X}^2(t,T) > 0. \quad (9)$$

Bonds are decreasing and convex in discrepancies between the actual and forecasted interest rates.

In order to evaluate the change in the whole value of assets and liabilities, for any realization of ΔK, we have to specify how - before setting up any reinsurance program - the premium P of the annuity is used for asset purchases. We assume that the type of bonds bought is determined by duration matching: the maturity T of the bonds is chosen so as to match the annuity one, i.e.

$$\tau^* = \frac{\sum_{u=t+1}^{\omega-x} u \times S_i(t,u)B(t,u)}{V_i^A(t)}$$

Given that bonds are zero-coupon, the asset duration, before reinsurance is bought, is $P(T-t)$. So, the two match if and only if

$$T^* = t + \frac{P}{\tau^*}. \tag{10}$$

Once the bond duration is identified, the fund may decide to transfer part of its longevity risk to a reinsurer, by paying a price ηC, $\eta \in [0,1]$. The part of the premium which is not used for transfer is left for bond purchases. The number of bonds bought is

$$n^* = \frac{P - \eta C}{B(t, T^*)}. \tag{11}$$

The financial risk incurred by the fund, as a consequence of this asset policy, can be evaluated at first order as follows. It is

$$\begin{aligned} &\left[-\Delta_A^F(t) + \frac{P-\eta C}{B(t,T^*)}\Delta_B^F(t,T^*)\right]\Delta K(t) = \\ &= \left[-\Delta_A^F(t) - \frac{P-\eta C}{B(t,T^*)}B(t,T^*)\bar{X}(t,T^*)\right]\Delta K(t) \\ &= \left[\sum_{u=t+1}^{\omega-x} B(t,u)S_i(t,u)\bar{X}(t,u) - (P-\eta C)\,\bar{X}(t,T^*)\right]\Delta K(t). \end{aligned} \tag{12}$$

At the same time, the fund is exposed to the part of demographic risk which it did not transfer. Approximating again the exposures to the first-order, the longevity risk of the fund is

$$(1-\eta)\Delta I_i(t) \sum_{u=t+1}^{\omega-x} B(t,u) S_i(t,u) X_i(t,u).$$

The expected financial return of the fund is

$$\begin{aligned} &\mathbb{E}_t\left[-V_A^F(t+dt) + V_A^F(t) + \frac{P-\eta C}{B(t,T^*)}\left[B(t+dt,T^*) - B(t,T^*)\right]\right] && (13) \\ \simeq\ &\mathbb{E}_t\left[\Delta K(t)\right]\left[\Delta_A^F + \frac{(P-\eta C)}{B(t,T^*)}\Delta_B^F\right] = && (14) \\ =\ &\mathbb{E}_t\left[\Delta K(t)\right]\left[\sum_{u=t+1}^{\omega-x} B(t,u)S_i(t,u)\bar{X}(t,u) - (P-\eta C)\,\bar{X}(t,T^*)\right]. && (15) \end{aligned}$$

Two extreme situations arise, when the reinsurance policy is either $\eta = 0$ or $\eta = 1$:

1. $\eta = 0$: the fund does not transfer demographic risk. It has financial risk from assets (bonds) and liabilities, since in this case $n^* = P/B$. Risks are

$$\Delta I_i(t) \sum_{u=t+t+1}^{\omega-x} B(t,u) S_i(t,u) X_i(t,u),$$

$$\Delta K(t)\left[\sum_{u=t+1}^{\omega-x} B(t,u)S_i(t,u)\bar{X}(t,u) - P\bar{X}(t,T)\right]. \tag{16}$$

while expected returns equal

$$\mathbb{E}_t\left[-V_A^F(t+dt)+V_A^F(t)+\frac{P}{B(t,T^*)}\left[B(t+dt,T^*)-B(t,T^*)\right]\right] \tag{17}$$

$$\simeq \mathbb{E}_t\left[\Delta K(t)\right]\left[\sum_{u=t+1}^{\omega-x} B(t,u)S_i(t,u)\bar{X}(t,u) - P\bar{X}(t,T)\right] \tag{18}$$

2. $\eta = 1$: the fund transferred all demographic risk. It has financial risk from assets, since $n^* = (P-C)/B$, and liabilities. This risk is equal to

$$\Delta K(t)\left[\sum_{u=t+1}^{\omega-x} B(t,u)S_i(t,u)\bar{X}(t,u) - (P-C)\,\bar{X}(t,T^*)\right]. \tag{19}$$

The expected returns of this strategy equal

$$\begin{aligned} & \mathbb{E}_t\left[-V_A^F(t+dt)+V_A^F(t)+\frac{P-C}{B(t,T^*)}\left[B(t+dt,T^*)-B(t,T^*)\right]\right] \\ \simeq\ & \mathbb{E}_t\left[\Delta K(t)\right]\left[\sum_{u=t+1}^{\omega-x} B(t,u)S_i(t,u)\bar{X}(t,u) - (P-C)\,\bar{X}(t,T^*)\right]. \end{aligned}$$

4 Trade-offs

In order to study the risk-return trade-offs, let us introduce the following notation:

$$\begin{aligned} \alpha &:= \sum_{u=t+1}^{\omega-x} B(t,u)S_i(t,u)X_i(t,u) > 0 \\ \beta &:= \sum_{u=t+1}^{\omega-x} B(t,u)S_i(t,u)\bar{X}(t,u) > 0 \\ \nu &:= \beta - (P-\eta C)\bar{X}(t,T^*) \\ \gamma &:= \beta - P\bar{X}(t,T^*) < \beta \\ \delta &:= \gamma + C\bar{X}(t,T^*) > \gamma \end{aligned}$$

So, for strategies 1 and 2 α is the Delta of the portfolio with respect to mortality risk, while γ and δ are the Deltas of the portfolios for the two strategies with respect to financial risk. Let C^* be the cost associated with the reinsurance policy. In the two strategies above we have $C^* = C$ and $C^* = 0$. The strategies are described in Table 1. Financial returns are evaluated at the end of the

Strategy	n^*	C^*	Dem risk	Fin risk	Net expected return
1	P/B	0	$\alpha\Delta I$	$\gamma\Delta K$	$\gamma\mathbb{E}\left[\Delta K\right]$
2	(P-C)/B	C	0	$\delta\Delta K$	$\delta\mathbb{E}\left[\Delta K\right] - C_{\Delta t}$

Table 1: Risks and expected return

interval Δt and are net of the costs $C^*_{\Delta t}$ of demographic-risk transfer, obtained as $\Delta t C^*/(\omega - x)$.[7]

Bonds partially offset the effect of any ΔK on liabilities. For instance, with $\Delta K < 0$, assets increase in value when γ and (a fortiori) δ are positive. The offsetting effect is larger when the whole amount of the premium P is used to buy bonds, under strategy 1.

In order to compare the two strategies - or any intermediate one, obtained by setting $\eta \in (0, 1)$ - we cannot use a standard risk/return frontier, for at least two reasons. First, we have not measured risk with any synthetic measure, such as the variance, yet. We have measured it through the change in the portfolio value - i.e. through the profit/loss - corresponding to every specific difference between forecasted and actual mortality ΔI or interest rate ΔK. We have a change in portfolio value for every couple or scenario (ΔI, ΔK). Second, we have two sources of risk, demographic and financial. As a consequence, we would need a risk/return frontier in three dimensions. In order to reconstruct a risk/return trade-off, without loosing the two sources of risk, in the next Section 4.1 we proceed in three steps. We first recognize the link between the scenario-based risk representation and a VaR risk-measurement for each risk factor. The second step consists in passing from the VaR of the factor to the VaR of the portfolio strategy. The third steps consists in summing up the VaRs due to financial and demographic risk to obtain the Overall VaR.

4.1 One-standard deviation shocks and VaRs

This section formalizes the move from risk factor changes to risk appraisal through VaR. We acknowledge that VaR is not a coherent risk measure, but we focus on it since it enters most current regulatory provisions. As an alternative, one could focus on Conditional Value at Risk (see for instance Cox et al. (2013)) or traditional mean-variance optimization (in a context similar to our, see the recent advances in Delong et al. (2008)).

Observe first that the expected values of the risk factors changes, $\Delta I_i = \Delta I_i(t+\Delta t)$ and $\Delta K = \Delta K(t+\Delta t)$ are equal to the expected values of the mortality intensity and interest rate, $\lambda_i(t+\Delta t))$ and $r(t+\Delta t)$, which we computed above, in (1) and (4), net of the corresponding forward rate. The variances

[7]Notice that we could subtract the whole cost of reinsurance – which lasts for the whole annuity maturity, $\omega - x$ – to compute financial returns. This would lower financial returns. The model can accomodate any splitting of the reinsurance cost over the maturity of the annuity.

are the ones computed in (2) and (5). So, using (1), (4), (2) and (5), we can compute $\mathbb{E}[\Delta I_i], \mathbb{V}ar[\Delta I_i], \mathbb{E}[\Delta K], \mathbb{V}ar[\Delta K]$.

A synthetic way to compare the two alternative strategies above consists in looking at what happens if both the mortality and interest rate over the next instant suffer a shock, i.e. have a realization which is not equal to their mean, $\mathbb{E}[\Delta I_i], \mathbb{E}[\Delta K]$. Each move $(\Delta I_i, \Delta K)$ different from $\mathbb{E}[\Delta I_i], \mathbb{E}[\Delta K]$, once substituted in the risk/return triples, provides a *scenario-based assessment of the strategy* itself. Consider first a positive or negative one-standard-deviation shock on the longevity of generation i and on interest rates:

$$\Delta I_i = \mathbb{E}[\Delta I_i] \pm 1 \times \sqrt{\mathbb{V}ar[\Delta I_i]}, \tag{20}$$

$$\Delta K = \mathbb{E}[\Delta K] \pm 1 \times \sqrt{\mathbb{V}ar[\Delta K]}. \tag{21}$$

It is straightforward to construct from this an evaluation of the VaR-type. Since both the intensity and the interest rate are Gaussian, indeed, looking at a one-standard-deviation shock means to examine the worst occurrence for I and K in 84% or 16% of the cases. Expressions (20) and (21) give the *VaR of the risk factors* at the level of confidence 84% - if we take $-1 \times \sqrt{\mathbb{V}ar[\Delta I_i]}$ - and 16%, if we take $+1 \times \sqrt{\mathbb{V}ar[\Delta I_i]}$. By changing the number of standard deviations examined - bringing it from 1 to 1.65 or 2.33, for instance - we would be looking at the worst scenarios for I and K in 95% and 5% or 99% and 1% of the cases. In general, we can fix a confidence level $1 - \epsilon$ (say 99, 95, 84) or ϵ at which the VaR of the risk factors can be evaluated, by choosing appropriately the constant in front of the standard deviation. Let $n(\epsilon)$ be that constant. So, the VaR of the two risk factors at the confidence level $1 - \epsilon$ is

$$VaR_{1-\epsilon}(\Delta I) = \mathbb{E}[\Delta I_i] - n(\epsilon)\sqrt{\mathbb{V}ar[\Delta I_i]}, \tag{22}$$

$$VaR_{1-\epsilon}(\Delta K) = \mathbb{E}[\Delta K] - n(\epsilon)\sqrt{\mathbb{V}ar[\Delta K]}. \tag{23}$$

However, in the end we are interested in the *VaR of the portfolio*, not in the VaR of the risk factors. According to Table 1, the realizations of the portfolio gains/losses are of the type $k\Delta I_i$ or $k\Delta K$, where the constant k can be either positive or negative ($k = \alpha, \beta, \gamma, \delta$). An increase in the risk factor corresponds to a portfolio loss if $k < 0$, to a gain if $k > 0$. This means that the VaR of the portfolio due to demographic risk is

$$kVaR_{1-\epsilon}(\Delta I_i) = k\left[\mathbb{E}[\Delta I_i] - n(\epsilon)\sqrt{\mathbb{V}ar[\Delta I_i]}\right] \qquad \text{if } k > 0, \tag{24}$$

$$kVaR_{\epsilon}(\Delta I_i) = k\left[\mathbb{E}[\Delta I_i] + n(\epsilon)\sqrt{\mathbb{V}ar[\Delta I_i]}\right] \qquad \text{if } k < 0. \tag{25}$$

Similarly for interest-rate risk. Table 2 reports the values of the financial and demographic VaR for each strategy.

Due to independency between financial and actuarial risk sources, if we sum up the appropriate scenario-based risks or VaRs (where appropriate stands for "based on the need of selecting VaR_ϵ versus $VaR_{1-\epsilon}$") we obtain the strategy-VaR due to both sources of risk. For each strategy we can compute the *overall*

Table 2: Demographic and Financial VaR for the four strategies

	Contribution to the strategy VaR	
Strategy	Demographic risk	Financial Risk
1	$\alpha VaR_{1-\epsilon}(\Delta I_i)$	$\gamma VaR_{1-\epsilon}(\Delta K)$ if $\gamma > 0$
	$\alpha VaR_{1-\epsilon}(\Delta I_i)$	$\gamma VaR_{\epsilon}(\Delta K)$ if $\gamma < 0$
2	0	$\delta VaR_{1-\epsilon}(\Delta K)$ if $\delta > 0$
	0	$\delta VaR_{\epsilon}(\Delta K)$ if $\delta < 0$

VaR, which we report in Table 3 together with the corresponding financial net expected return. This opens the way to representing the trade-offs of the strategies in a familiar way, by associating to each strategy a point in the plane (Overall-VaR, net expected return).

Table 3: Overall VaR for strategies 1 and 2

Strategy	**(VaR, expected return) combination**
1	$(\alpha VaR_{1-\epsilon}(\Delta I_i) + \gamma VaR_{1-\epsilon}(\Delta K), \gamma \mathbb{E}[\Delta K])$ if $\gamma > 0$
	$(\alpha VaR_{1-\epsilon}(\Delta I_i) + \gamma VaR_{\epsilon}(\Delta K), \gamma \mathbb{E}[\Delta K])$ if $\gamma < 0$
2	$(\delta VaR_{1-\epsilon}(\Delta K), \delta \mathbb{E}[\Delta K] - C_{\Delta t})$ if $\delta > 0$
	$(\delta VaR_{\epsilon}(\Delta K), \delta \mathbb{E}[\Delta K] - C_{\Delta t})$ if $\delta < 0$

5 Comparing strategies

In order to state whether the risk transfer is worthwhile and to what extent, we need to compare all the possible strategies: the limit strategies 1 and 2 described above and the ones in which demographic risk is partially reinsured. In general, we can describe the triples (Demographic Risk, Financial Risk, Net Expected Return) for each strategy as:

$$((1-\eta)\alpha \Delta I, \nu \Delta K, \nu \mathbb{E}[\Delta K]).$$

We compare these competing strategies in the plane (Overall-VaR, Expected Financial Return) (for simplicity, we skip the term "net" from now on, even though costs are net of the fraction of transferring costs, if applicable). Hence, the comparison is between the strategies obtained with different values of η:

$$\begin{cases} ((1-\eta)\alpha VaR_{1-\epsilon}(\Delta I_i) + \nu VaR_{1-\epsilon}(\Delta K), \nu \mathbb{E}[\Delta K]) & \text{if } \nu \geq 0, \\ ((1-\eta)\alpha VaR_{1-\epsilon}(\Delta I_i) + \nu VaR_{\epsilon}(\Delta K), \nu \mathbb{E}[\Delta K]) & \text{if } \nu < 0. \end{cases} \tag{26}$$

For any given confidence level ϵ for VaR, the set of ALM strategies for different values of η is represented by a kinked line between 1 and 2 (Figure 1). The Delta of the portfolio with respect to demographic risk is always positive ($\alpha > 0$). As long as η increases and strategies move from strategy 1 towards 2, demographic risk decreases proportionally. The kink of the line corresponds to the point at which the Delta of the portfolio of assets and liabilities – i.e. short the annuity and long the bond – with respect to the financial risk is null. This point corresponds to the value η which solves:

$$-\Delta_A^F(t,T) - (P - \eta C)\bar{X}(t,T^*) = 0. \tag{27}$$

The reinsurance level η which satisfies this condition can be simply evaluated. The kink exists since the financial component of the overall VaR is computed considering one or the other queue of the distribution of ΔK according to the sign of ν, as (26) shows. Exposure to financial risk increases is increasing in η. As long as we move to the left of the kink, the line is negatively sloped, since both the demographic and financial risks decrease with η. To the right of the kink, the line can be negatively or positively sloped, since reinsuring demographic risk decreases the corresponding VaR, while financial risk is increasing. If the second effect prevails, the line is positively sloped and the corresponding frontier is made by dominated strategies (for any level of risk, there is a combination of assets and liabilities with a higher return). This in turn happens as a combination of the magnitude of the coefficients in Table 1, together with the fact that in VaR only the "worst possible outcome" of the corresponding factor matters (the queue can be the left or right one).[8] In this case, the part of the frontier for which $\nu < 0$ is suboptimal, since its points represent risk-return combinations which are dominated. For each of them one can find a reinsurance quota which gives the same overall VaR and a higher expected return.

Let us now describe a decision criterion for the optimal reinsurance policy of the fund. We define the risk-return preferences of the fund through a utility function defined on the plane (Overall VaR, Expected Return). This choice does not pretend to be axiomatically based, but simply to be consistent with a regulatory, VaR-based measurement of risk. Given these preferences, we can choose $\eta^* \in [0,1]$ which maximizes an expected utility of the type:

$$U(\mu, VaR_\epsilon(\Delta K, \Delta I), \xi), \tag{28}$$

with $U' > 0, U'' < 0$, where ξ is a parameter (or, possibly, a set of parameters) describing the risk attitude of the fund. Graphically, the best strategy is identified as the point of the frontier that crosses the highest possible indifference curve, as represented in the figures. The point identifies the optimal level of reinsurance demanded by the fund.

[8] The reason why there is still financial risk left, in spite of duration matching, is that the duration itself is a classical one, not the Delta or Delta-Gamma duration matching which can be performed in the Hull-White setting (see for instance Avellaneda (2000)). The latter one would eliminate any riskiness up to first or second order approximations, but would leave no room for exploiting the risk-return trade-off, i.e. for optimization.

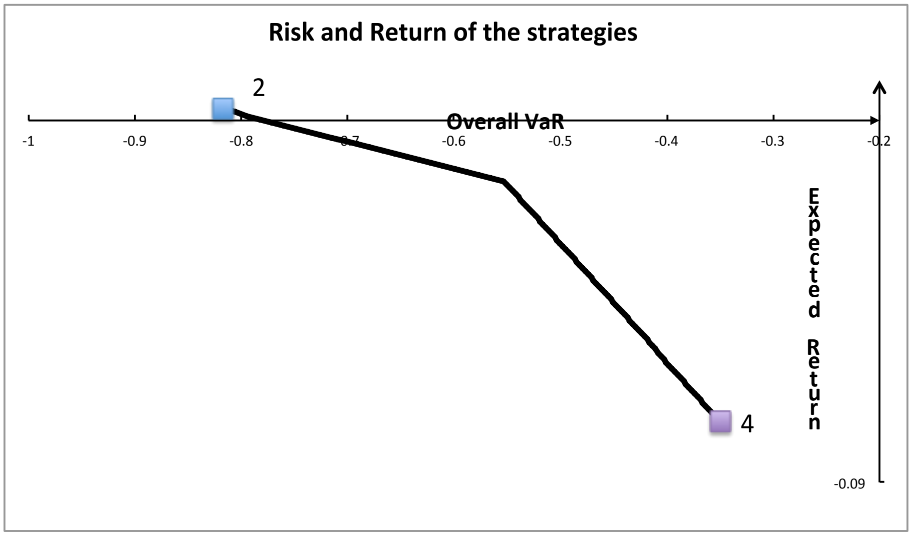

Figure 1: This figure shows the risk-return combinations of the set of strategies 1 and 2, for all the possible values of η, when $n^* > 0$ and financial risk does not increase when reducing demographic risk. The strategies are represented by the black broken line. The dotted curve is the highest indifference curve which is tangent to the set of strategies. The optimal strategy lies at the intersection between the curve and the line. On the horizontal axis the VaR at a certain level ϵ is reported, while the expected financial return net of reinsurance costs lies on the vertical axis.

In order to appreciate the trade-offs so elicited and the different informations we can convey on them, we introduce and comment an example calibrated on UK mortality and financial data.

6 Implementing and comparing strategies on UK data

We implement the strategies designed above and study their trade-offs using data from the UK market. To be specific, we consider a whole-life annuity sold on a UK male head aged 65 at strategy inception, December 30, 2010; we take financial data from the UK Government market on the same date. We then presume that the revenues from annuity sales are invested in UK Government bonds. The Hull-White model is then calibrated to zero-coupon bond prices at the same date. Under the risk-neutral measure its parameters are $g = 2.72\%$, $\theta = 17.23\%$, $\Sigma = 0.65\%$, while $r(0) = 0.42\%$. The market price of risk (5%) is chosen so that the long-run mean under the historical measure is around 4%. The survival rates are calibrated from projected IML92 tables. For the generation we considered, the model parameters are $a_i = 10.94\%, \sigma_i = 0.07\%$ and $\lambda_i(0) = 0.885\%$. Table 4 summarizes all the relevant parameters. Table 5

Table 4: Calibrated parameters

Symbol	Value
Financial risk	
g	2.72%
Σ	0.65%
θ	17.23%
r(0)	0.42%
Demographic risk	
a_i	10.94%
σ_i	0.07%
$\lambda_i(0)$	0.885%

reports the prices and the Deltas of the instruments we use in the example.

Table 5: Risk exposures and prices of instruments

Figure	Symbol	Value
Annuity		
Price	V^A	13.09
Exposure to longevity risk	Δ_A^M	-323.48
Exposure to financial risk	Δ_A^F	-100.92
10-year bond		
Price	B(0,9.19)	0.786
Exposure to financial risk	Δ_B^F	-6.387

The fair price of the annuity − which is also its selling price − is $V^A = P = 13.09$.[9] Being short the annuity, which has exposures $\Delta_A^M = -323.48$ and $\Delta_A^F = -100.92$ the fund remains exposed positively to both risk factors change. The fund operates on the financial market using a bond whose maturity is computed according to (10) and is $T^* = 9.19$. We assume the existence of such a bond, which is priced $B(0, 9.19) = 0.786$ and has $\Delta_B^F = -6.387$. We evaluate the hedging strategies we described above at an horizon $\Delta t = 1$ year. The longevity risk factor $I(1)$ is expected to be positive, $\mathbb{E}_0[I(1)] = 2.73 * 10^{-7}$ while its variance is $\mathbb{V}ar_0[I(1)] = 5.47 * 10^{-7}$. Demographic risk can be transferred to a reinsurer at its fair price $C = 2.95$. The expected value of the financial risk factor under the historical measure is slightly negative, equal to $\mathbb{E}_0[K(1)] = -0.04\%$, while its variance is $\mathbb{V}ar_0[K(1)] = 3.79 * 10^{-5}$. We charge

[9] In the computation, we considered $\omega = 110$ years.

expected returns not with the whole reinsurance cost, but with the part which refers to the cover of the horizon considered. As a consequence, financial returns are $\mathbb{E}_0[K(1)] - C_{\Delta t}$ and $C_{\Delta t} = 0.065$.

The coefficient δ is positive, 18.47, while γ is negative, -5.52. In Table 6 we report the exposures, the expected financial return net of the reinsurance cost and the remaining liquidity of the two strategies.

Table 6: Risk exposures, VaR and expected returns

Figure		Strategy	
	Symbol	1	2
Optimal number of bonds	n^*	16.66	12.90
Optimal cost of reinsurance	C^*	0	2.95
Exposure to longevity risk	$\alpha/0$	323.48	0
Exposure to financial risk	γ/δ	-5.52	18.47
Liquidity	$P - C^* - n^*B$	0	0
Expected financial return	μ	-0.063	-0.075
$VaR_{99.9\%}$ demographic risk	$VaR_{99.9\%}(\Delta I)$	-0.72	0
$VaR_{99.9\%}$ financial risk	$VaR_{99.9\%}(\Delta K)$	-0.10	-0.35
Overall $VaR_{99.9\%}$	$VaR_{99.9\%}$	-0.82	-0.35
Overall $VaR_{84\%}$	$VaR_{84\%}$	-0.27	-0.12

Both strategies make use of all the available resources P received as a premium for the annuity, and no liquidity is left.

Strategy 1 invests all P in $n^* = 16.66$ bonds. It offers a positive expected return, 0.003, since the fund has negative exposure to financial risk ($\Delta^F = -5.52$) and the expected value of the risk factor is negative too. The overall VaR, which is computed at a one-year horizon at a 99.9% confidence level, is -0.81 and it is due mostly to longevity risk (-0.71). Strategy 2 hedges against longevity risk and invests the remaining resources $P - C$ to buy $n^* = 12.90$ bonds. The overall VaR is reduced from -0.96 of strategy 1 to -0.35 on the one side, but expected financial returns are lower (-0.075 vs. 0.007), mainly because of the cost of reinsurance which is paid for the longevity risk transfer.
Figure 2 represents the trade-off between VaR and expected return in place between these two competing strategies. The line representing their trade-off is negatively sloped, in all of its parts. We do not have financial risk offsetting the decrease in demographic risk, when the latter is reduced. The fund accepts to deal with a higher risk to obtain a higher expected return.

If we specify a utility function for the fund, defined with respect to expected returns and overall VaR, the fund can optimally choose between the competing strategies the strategy that maximizes utility. Let us consider for example a

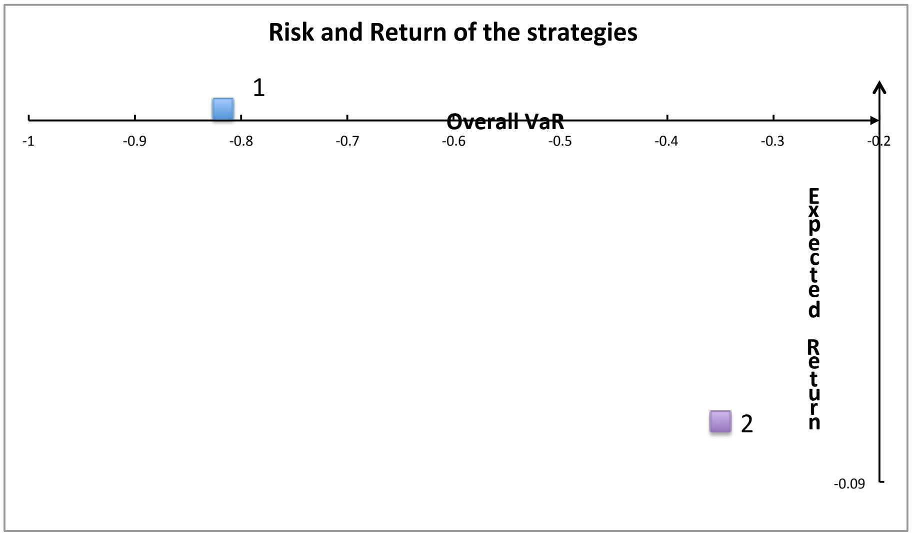

Figure 2: This figure shows the risk-return combinations of strategies 1 and 2 for the UK case. On the horizontal axis the VaR at level 99.9% is reported, while the expected financial return net of reinsurance costs lies on the vertical axis.

simple expected utility function

$$U(\mu, VaR) = \mu - \frac{(VaR_{99.9\%})^2}{\xi},$$

in which $\xi > 0$ is a measure of risk aversion correlated with the risk aversion coefficient.
Let us set it to $\xi = 5$. Comparing the utility of the two strategies, the fund, which is highly risk averse would choose strategy 2, for which U = -0.099, over strategy 2, for which U = - 0.131. We now allow the fund to reinsure a part $\eta \in [0, 1]$ of its longevity exposure in order to maximize its utility. Figure 3 represents the set of possible strategies, which is the line between 1 and 2, and the highest indifference curve that crosses this set. The tangency point determines the optimal fund strategy. In the figure, the dotted line represents the highest possible indifference curve which is tangent to the set of possible fund strategies. The optimal strategy for this fund implies reinsuring $\eta = 23.02\%$ of the longevity exposure (at a total cost of 0.68, of which 0.0015 imputed to the first year of the contract) and buying 15.80 bonds. Notice that this optimal strategy has no exposure to financial risk. The optimal strategy is then characterized by

$$U^* = -0.076$$

which is higher than the one of strategy 2, a

$$VaR = -0.55$$

and an instantaneous expected return $\mu = -0.015$, which is entirely due to the reinsurance cost.

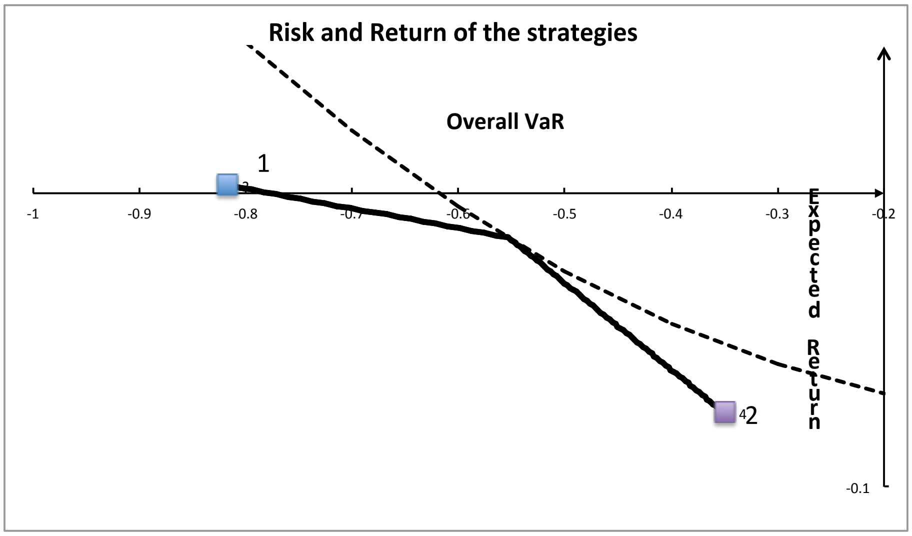

Figure 3: This figure shows the risk-return combinations of the set of strategies (the broken line between 2 and 4) once partial longevity reinsurance is admitted. On the horizontal axis the VaR at level 99.9% is reported, while the expected financial return net of reinsurance costs lies on the vertical axis. The dotted line represents the highest possible indifference curve of the utility function that crosses the set of strategies.

Even in the UK case, though, if ever the transfer price of longevity risk increases, transferring longevity risk may increase the overall VaR (in absolute value), since the increase in the financial component may become larger than the decrease in the demographic one. This is the case which cannot be captured by those approaches which do not take a holistic view of risk. For instance, if the transferring price (instead of being fair as above) becomes as high as 5, then the risk exposures, VaRs and returns in correspondence of the pure strategies 1 and 2 are represented in Table 7.

The corresponding frontier is represented in Figure 4.

The straight line connecting strategies 1 and 2 becomes positively sloped when reinsurance increases. Due to the high cost of the transfer, indeed strategy 2 allows to buy a small number of bonds, 10.28, whose exposure offsets just a small part of the financial risk due to the annuity position. Starting from the pure strategy 1, with no reinsurance, and increasing it, we first reduce the overall risk (in absolute value), even though we decrease the expected return. Beyond the kink, which is the point where the exposure to financial risk of the whole portfolio (asset and liabilities) is zero, though, VaR increases again (this is the lower part of the frontier), while expected return continues to go down, since

Table 7: Risk exposures, VaR and expected returns when the cost of the transfer increases

Figure		Strategy	
	Symbol	1	2
Optimal number of bonds	n^*	16.66	10.28
Optimal cost of reinsurance	C^*	0	5
Exposure to longevity risk	$\alpha/0$	323.48	0
Exposure to financial risk	β/β	-5.52	35.26
Liquidity	$P - C^* - n^*B$	0	0
Expected financial return	μ	0.003	-0.129
$VaR_{99.9\%}$ demographic risk	$VaR_{99.9\%}(\Delta I)$	-0.72	0
$VaR_{99.9\%}$ financial risk	$VaR_{99.9\%}(\Delta K)$	-0.10	-0.67
Overall $VaR_{99.9\%}$	$VaR_{99.9\%}$	-0.82	-0.67

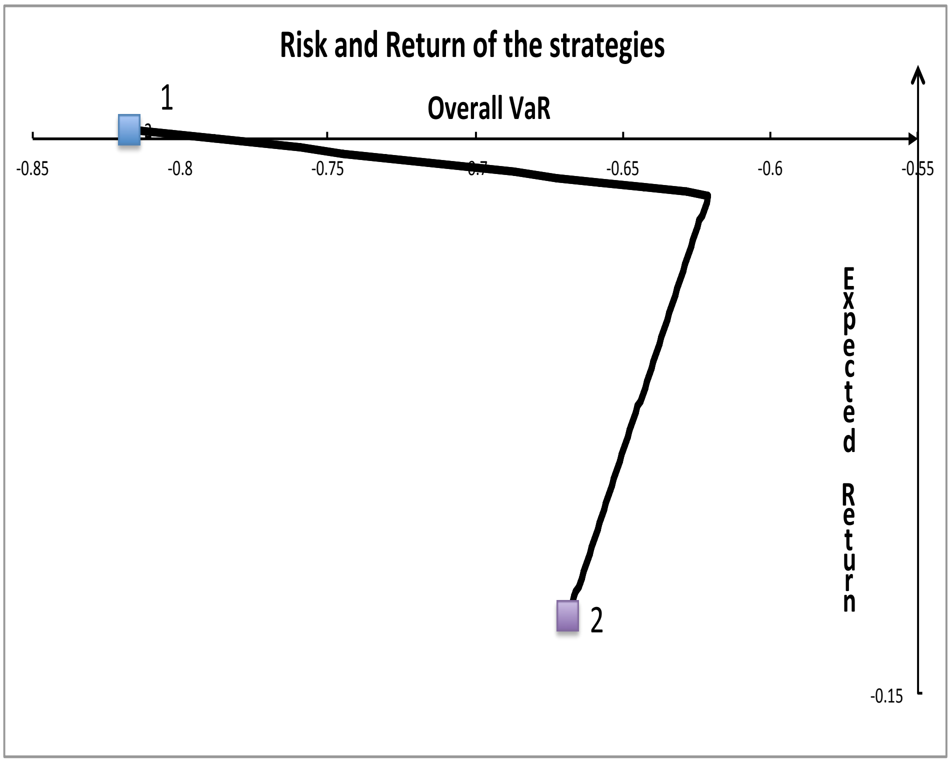

Figure 4: This figure shows the risk-return combinations of the set of strategies (the broken line between 1 and 2) once partial longevity reinsurance is admitted, when the cost of the transfer is C=5. On the horizontal axis the VaR at level 99.9% is reported, while the expected financial return net of reinsurance costs lies on the vertical axis.

financial risk increases so much that it offsets the relief in demographic risk. All the combinations of risk and return on this part of the frontier are clearly suboptimal. We could not have captured this effect by looking only at financial or longevity risk, or modelling them separately and differently. This is why we consider highlighting this effect an important - if not the main - contribution of this paper.

7 Conclusions and related research

This paper explores the risk-return trade-off of a pension fund when it is possible to transfer longevity risk. Our primary goal was to take a holistic view of financial and longevity risk management, since demographic risk transfer impacts on interest-rate risk exposure. We assessed risk through Value-at-Risk from both financial and longevity shocks. The optimal transfer choices of the fund are located along the (VaR, Expected Return) frontier.

The main result is that transfer of longevity risk may decrease or increase VaR in absolute value, since it decreases its longevity part, but may either increase or decrease the financial one. So, the frontier can be positively or negatively sloped. This is the case which cannot be captured by those approaches which do not take a holistic view of risk. We provide a fully calibrated example in which the frontier can indeed be negatively sloped.

Our approach puts into evidence the main counterintuitive effect of reinsurance. In order to do so, we take a simplified view of the problem. We cover a single annuity, which stands for a homogeneous group of them. On the asset side we allow only for risky bonds and cash. Reinsurance is priced starting from its hedge on the part of the reinsurer, while longevity transfer through more complex derivatives is not formalized. Last, we concentrate on a single generation (as Delong et al. (2008) and Cox et al. (2013) do) and disregard minimum capital requirements. All the realistic features, such as a richer liabilities portfolio with idiosyncratic risk or a richer investment opportunity set, or more complex liability-risk transfer, are left for future extensions. Multiple-generation versions are an obvious extension. Another straighforward extension consists in comparing VaR with its capital absorption. We indeed focused on VaR since most of the current regulation uses it for risk appraisal. If we link – as we could easily do – our VaR measure to the capital requirement of the fund, the capital absorption and its cost enter our picture in quite a straightforward way. This would permit us to extend the analysis to the case in which the fund is subject to regulatory capital requirements. The objective of the fund would then be to maximize its utility, subject to a solvency constraint.

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