Deductions for Early Retirement

Markus Knell*

Oesterreichische Nationalbank

Economic Studies Division

February 2017

Abstract

The paper studies how the rates of deduction for early retirement have to be determined in PAYG systems in order to keep their budget stable. I show that the budget-neutral deductions depend primarily on the specific rules of the pension system and on the demography while the market interest rates are less influential. In fact, if the distribution of retirement ages is stable over time then the market rates are completely irrelevant. In non-stationary situations the system has to finance a deficit and the market interest rates play a role. Even then, however, their importance is smaller than in the benchmark framework that is typically used in the related literature. For realistic demographic assumptions it is sufficient to use a discount rate that is close to the implicit rate of return of the PAYG system in order to calculate the actuarial deductions.

Keywords: Pension System; Demographic Change; Financial Stability;
JEL-Classification: H55; J1; J18; J26

*OeNB, Economic Studies Division, Otto-Wagner-Platz 3, POB-61, A-1011 Vienna; Phone: (++43-1) 40420 7218, Fax: (++43-1) 40420 7299, Email: Markus.Knell@oenb.at. The views expressed in this paper do not necessarily reflect those of the Oesterreichische Nationalbank.

1 Introduction

This paper discusses how pay-as-you-go (PAYG) pension systems have to determine actuarial deductions for early retirement and supplements for late retirement in order to remain financially balanced in the long run. Deductions are necessary since an insured person who retires at an earlier age pays less contributions into the pension system than an otherwise identical individual and he or she also receives more instalments of (monthly or annual) pension payments.[1] Despite the fact that the levels of deductions are thus crucial parameters for pension design that are present in all real-world systems the existing literature on this issue is rather small and at times even confusing. In this paper I use a simple model to discuss this topic in a systematic and transparent manner.

It is helpful to clarify right at the beginning how the notion of "actuarial neutral deductions" is used in this paper. There are three issues that have to be scrutinized. The first important issue concerns the distinction between deductions and reductions. PAYG systems are organized according to different principles. Many systems are based on a target retirement age R^* and on pension rules that specify a formula pension $\widehat{P}(R)$ for retirement at an arbitrary age R. Typically for $R < R^*$ this formula pension $\widehat{P}(R)$ will be lower than the target pension level $P^* = \widehat{P}(R^*)$ but larger than the pension level $P(R)$ that is compatible with a balanced budget. In this case one needs additional deductions applied to the formula pension in order to preserve financial stability. I call the total adjustment from the target pension to the budget-neutral pension the *pension reduction* ($P(R)/P^*$) and the adjustment of the formula pension to the budget-neutral pension the *pension deduction* ($P(R)/\widehat{P}(R)$).

Below I will discuss three variants of PAYG systems. First, a generic defined benefit (DB) system that is just defined by the target pension level P^* promised at the target retirement age R^*. In this case the necessary reduction for early retirement is entirely implemented via the deduction. Second, an accrual rate system (as it is, e.g., in place in Germany or France) where the basic pension formula already implies a lower pension payment for earlier retirement. In this case part of the necessary total reduction is thus implemented via the pension formula while the remaining part is implemented via the deduction. Third, I also look at the popular notional defined contribution (NDC) system (introduced in countries like Sweden, Italy or Poland). This system mimics a funded

[1] From now on I will focus on the case of early retirement and the associated deductions. All of the following statements and results, however, also hold for the opposite case of late retirement and associated supplements.

system and it does not rely on a target retirement age R^* or a target pension level P^*. It rather uses a formula that adjust pension payments to the fact that early retirement is associated with fewer years of contributions *and* with more years of pension payments. Under certain conditions a NDC system does not need additional deductions and the entire necessary reduction is generated by the pension formula.

The second important issue is the clarification of the perspective, i.e. the answer to the question: "neutral for whom"? In this paper I focus on "actuarial neutrality from the viewpoint of the pension system". This means that for the system it should not matter whether an individual retires at the target retirement age R^* or whether he/she retires before or after that age. This system-wide or "macroeconomic" viewpoint of actuarial neutrality has to be distinguished from the "microeconomic" or "incentive-compatible" viewpoint (cf. Börsch-Supan 2004, Queisser & Whitehouse 2006). The latter approach focuses on the determination of deductions that leave an *individual* indifferent between retiring at the target age or an alternative age. This individual perspective is often used to analyze incentives of existing pension systems for early or delayed retirement (Stock & Wise 1990, Gruber & Wise 2000, Shoven & Slavov 2014, Gustman & Steinmeier 2015). "Actuarial neutrality" must also be distinguished from the notion of "actuarial fairness". The latter concept is often used to describe a system in which the present value of expected contributions is ex-ante equal to the present value of expected pension payments (cf. Börsch-Supan 2004, Queisser & Whitehouse 2006). This is therefore a life-cycle concept while actuarial neutrality can be regarded as a "marginal concept" that focuses on the effect of postponing retirement by an additional year. The different concepts are related to each other but they are not identical and they can lead to different conclusions. For sake of clarity I will therefore often use "budget neutral deductions" in order to refer to the notion of "actuarial neutral deductions from the viewpoint of the pension system".

The third issue concerns the reference point of budgetary neutrality, i.e. the answer to the questions: "neutral as compared to what"? In fact, the budgetary implications of a single retirement decision cannot be evaluated without knowing or making assumptions about the retirement decisions of all other current and future retirees. An individual choosing an early retirement age $R^L < R^*$ will cause a shortfall if all other individuals retire at the target age. In contrast to this one-time shock assumption, the early retirement age of some might, however, also be counterbalanced by later retirement $R^H > R^*$ of others without causing any budgetary imbalances. The budget-neutral deductions thus can only be calculated relative to an assumption concerning the *collective* behavior. This important point has so far been almost neglected in the related literature that has primarily focused on the one-time-shock-scenario described above.

In the first part of the paper I use a simple demographic model that allows for analytical solutions. In a later part it is shown that all of the results also hold in a more general model with an arbitrary mortality structure. Throughout the paper I abstract from interpersonal differences in wages and life expectancy in order to concentrate on the main issues. For the same reason I also abstract from disability and survivor benefits and focus on old-age pensions only. The starting point of the analysis is the equation that captures the neutrality between retiring at the target age R^* and retiring at an earlier age $R < R^*$. For the simple model this equation can be solved for the level of deductions X that guarantees budgetary neutrality. In a next step I compare the deduction factors for three specific pension formulas: the DB, the AR and the NDC system. It comes out that for each system the deduction factor X_j for $j \in \{\text{DB}, \text{AR}, \text{NDC}\}$ can be expressed as the product of a "demographic part" Ψ_j that just depends on demographic and retirement variables and a "financing part" Δ that also depends on the discount rate δ. The financing part is the same for all three systems and it is equal to $\Delta = 1$ if the discount rate is the same as the implicit rate of return of the PAYG system. The demographic adjustment factor, on the other hand, differs among the three systems. It is $\Psi_{NDC} = 1$ for the NDC system which means that the basic formula of the NDC system is sufficient to implement the required reduction for early retirement as long as the discount rate is chosen such that $\Delta = 1$. For the other two systems this is not true and the demographic deduction factors Ψ_{DB} and Ψ_{AR} have to be used in order to safeguard financial stability. Numerical examples illustrate the size of the annual budget-neutral deductions for different pension systems, different retirement ages and different discount rates. For the DB system they range between 5.9% and 11.8%, for the AR system between 4.2% and 9.8% and for the NDC system between 0% and 4.9%. The size of the necessary deductions for the DB and the AR systems is mainly driven driven by the demographic part while the financing part is of lesser quantitative importance.

The choice of the discount rate is nevertheless a crucial factor in determining the size of the appropriate deductions. As stated above this choice depends on the assumption concerning the collective retirement behavior. If the retirement age follows a stationary distribution over time then one can derive the following results. First, a NDC system leads to a stable budget without the use of additional deductions or supplements. Second, the DB and AR systems are also compatible with balanced budgets if they are augmented by the demographic deduction factors. Third, the discount rate that corresponds to these budget-neutral demographic deduction factors is given by the implicit rate of return of the

PAYG system. Fourth, the choice of a higher discount rate might still be associated with a balanced budget if the target retirement age R^* is equal to the average actual retirement age $\overline{R}$. If this is not the case then the system will run permanent surpluses or deficits. A situation like this looks problematic from the viewpoint of interpersonal distribution since it implies that early retirees have to pay deductions that are larger than what is necessary for budgetary stability while late retirees are offered larger-than-necessary supplements.

The situation of a stationary retirement distribution is certainly a stylized benchmark and I also study the case of non-stationarity. I discuss a reasonable approach to deal with this situation. In particular, I assume that the system can define a target distribution of retirement ages and compare the actual distribution in a certain year to this target distribution. The time-dependent rates of deduction are related to the additional financing needs that are due to the deviation from the actual to the target distribution. The target distribution might be given by the distribution of the past or by some "normative" distribution. The one-time-shock scenario where everybody except one individual is assumed to retire at the target age R^* is only one (rather extreme) example of such a (normative) target distribution. I illustrate the numerical implications for more modest target distributions and the ensuing rates of deductions are typically considerably lower and much closer to the case of stationarity.

Overall, the findings of the paper suggest that the size of budget-neutral deductions are mainly influenced by the rules of the pension system (like the target retirement age and the pension formulas) and the demographic structure (like remaining life expectancy). The market interest rate is irrelevant for stationary situations and of minor importance for realistic non-stationary developments of the retirement age. The use of higher rates of deduction that are sometime suggested for reasons of incentive compatibility seem problematic from this perspective since they involve excessive punishments for early retirement and excessive rewards for later retirement with probably unindented intragenerational redistributions.

There exists a broad literature on actuarial adjustments from the perspective of the insured individuals. Papers in this branch of the literature have studied the impact of non-actuarial adjustments on early retirement (Gruber & Wise 2000), the incentives to delay retirement (Coile et al. 2002, Shoven & Slavov 2014) and the simultaneous decisions on retirement, benefit claiming and retirement in structural life cycle models (Gustman & Steinmeier 2015). The literature on actuarial adjustments from the perspective of the pension system—which is the focus of this paper—is, however, rather small. Queisser & Whitehouse (2006) offer terminological discussions and they present evidence on the observed rates of adjustments. For a sample of 18 OECD countries they report an average annual deduction for early retirement of 5.1% and an average annual supplement of 6.2% for late retirement. They conclude that "most of the schemes analysed fall short of actuarial neutrality [and that] as a result they subsidise early retirement and penalise late retirement" (p.29). An intensive debate about this topic can be observed in Germany where the rather low annual deductions rates of 3.6% are regularly challenged. A number of researchers have supported higher rates of deductions based on market interest rates (Werding 2007) while other participants have argued for keeping rates low stressing the lower implicit rate of return of the PAYG system (Ohsmann et al. 2003). Overviews of the debate can be found in Börsch-Supan (2004) and Gasche (2012).

The paper is organized as follows. In section 2 I focus on the simple model that allows for analytical solutions. In section 3 I analyze the cases of a stationary retirement distribution in a general mortality setting while in section 4 I focus on a non-stationary situation. Section 5 discusses some possible extensions and section 6 concludes.

2 Simple framework

2.1 Set-up

I start with the benchmark approach of the related literature. In order to fix ideas I focus on the most simple case with a constant wage W and a stable demographic structure where all individuals start to work at age A, are continuously employed and die at age ω.[2]

There exists a PAYG pension system with a constant contribution rate τ, a target (or reference) retirement age R^* and a pension formula that determines the regular pension for each admissible retirement age R. In the most simple form the system only determines the pension level P^* that is promised for a retirement at the target age R^*. In this case the pension deductions are the only instrument to implement appropriate adjustments for early retirement. Many real-world pension systems, however, are based on a "formula pension" that depends on the target retirement age R^* *and* on the actual retirement age R thereby accounting (at least partially) for early retirement. This formula pension is denoted by $\widehat{P}(R, R^*)$ and I will discuss below various possibilities for its determination. For the moment, however, I leave it unspecified. Furthermore, for brevity I will often omit

[2] In section 3 I will discuss the case where wages grow at rate $g(t)$ and where there exists mortality before the maximum age ω.

the arguments of $\widehat{P}(R, R^*)$ and other functions whenever there is no risk of ambiguity. Finally, the pension level at the target retirement age is given by $P^* \equiv \widehat{P}(R^*, R^*)$.

2.2 Deductions for a general discount rate

If an individual chooses to retire before the target age (i.e. $R < R^*$) then the actuarial neutral deduction for early retirement will reduce the formula pension payment $\widehat{P}$ in such a way that the retirement decision has no long-run effect on the budget of the social security system. In order to do so two effects have to be taken into account. First, for the periods between R and R^* the individual does not pay pension contributions and thus the system has a shortfall of revenues. Second, in these periods of early retirement the individual already receives pension payments and thus the system has to cover additional expenditures. The formula pension level $\widehat{P}$ thus has to be reduced by a factor X (that is valid for the *entire* pension period) in order to counterbalance these two effects. The final pension will thus be given by $P = \widehat{P}X$. Using a continuous time framework the actuarial deduction factor X is implicitly defined as follows:

$$\int_R^{R^*} \left(\tau W + \widehat{P}X\right) e^{-\delta(a-R)}\,\mathrm{d}a = \int_{R^*}^{\omega} \left(P^* - \widehat{P}X\right) e^{-\delta(a-R)}\,\mathrm{d}a, \qquad (1)$$

where δ is the social discount rate used to evaluate future payment streams. The left-hand side of equation (1) contains the twofold costs to the system due to early retirement (i.e. the period loss of contributions $\tau W$and the additional period expenditures $\widehat{P}X$). The right-hand side captures the benefits to the system since in the case of a retirement at the target age the pension without deductions would be P^* for all periods between R^* and ω which is now reduced to $P = \widehat{P}X$.[3]

The determination of the discount rate δ is a crucial issue. In fact, it will turn out that different approaches to calculate appropriate deductions differ primarily in their choice of the discount rate (see also Gasche 2012). If the costs of early retirement have to be financed by debt then the market interest rate seems to be the right choice, i.e. $\delta = r$. If, one the other hand, the budget of the system remains subdued (e.g. because early retirement of some is counterbalanced by late retirement of others) then a lower interest rate like the implicit rate of return of the PAYG system seems appropriate.

A different way to derive equation (1) is based on the well-known concept of "social

[3] The same logic also holds for late retirement with $R > R^*$. In this case the equivalent to (1) is given by: $\int_{R^*}^{R} \left(\tau W + P^*\right) e^{-\delta(a-R)}\,\mathrm{d}a = \int_{R}^{\omega} \left(\widehat{P}X - P^*\right) e^{-\delta(a-R)}\,\mathrm{d}a$. This can be transformed to yield (1).

security wealth" (SSW) which will be useful later. The SSW corresponds to the present value at some planning age S of the expected discounted pension benefits minus the expected contribution payments (see Stock & Wise 1990). The SSW depends on the planning age S, the retirement age $R > S$, the pension system (captured by $\widehat{P}$ and X) and on the discount rate δ. One can thus write:

$$SSW(S,R,X,\delta) = \int_R^\omega \widehat{P} X e^{-\delta(a-S)}\,\mathrm{d}a - \int_S^R \tau W e^{-\delta(a-S)}\,\mathrm{d}a. \qquad (2)$$

For retirement at the target retirement age R^* there are no deductions (i.e. X=1). The idea of actuarial neutrality can thus be expressed by the condition $SSW(S,R^*,1,\delta) = SSW(S,R,X,\delta)$. It is straightforward to show that this expression is equivalent to equation (1). On the other hand, one can define "actuarial fairness" as a situation where the pension system provides ex-ante net payments of zero, independent of the retirement age R, i.e. $SSW(A,R,X,\delta) = 0$ (see Börsch-Supan 2004, Queisser & Whitehouse 2006).[4]

One can solve equation (1) for X which gives rise to a rather complicated expression. Linearization of this result (around $\delta = 0$) leads to the approximated value $\widetilde{X}$:

$$\widetilde{X} = \frac{\omega - R^*}{\omega - R}\left[\frac{P^*}{\widehat{P}} + \frac{\tau W}{\widehat{P}}\frac{R-R^*}{\omega - R^*} + \frac{\delta}{2}(R-R^*)\left(\frac{P^*}{\widehat{P}} + \frac{\tau W}{\widehat{P}}\right)\right]. \qquad (3)$$

2.3 Different PAYG systems

In order to further evaluate expression (3) and to derive numerical values one has to specify how the formula pension level $\widehat{P}$ is determined. There exist various possibilities and I will discuss three variants that are regularly used in existing pension systems.

- **Defined Benefit (DB) System**: In this case there exists a target replacement rate q^* that is promised if an individual retires at the target retirement age R^*. In the generic DB case the pension formula is independent of the actual retirement age and does not reduce the target replacement rate, i.e.: $\widehat{P}_{\mathrm{DB}}(R,R^*) = q^* W$.

[4]Note that equation (1) could also be viewed as the condition that makes an individual indifferent between retirement at the age of R^* and retirement at an earlier age R. In this case the appropriate discount rate is given by his or her individual rate of time preference which is typically assumed to be given by the market interest rate. The deductions derived under this approach are sometimes called "incentive compatible" which corresponds to "actuarial neutrality" from the perspective of the insured person. The paper, however, focuses on "budget neutral" deductions which corresponds to "actuarial neutrality" from the perspective of the insuring system. The individual SSW is often used in the context of the incentive-compatible approach in order to study the incentives for early or delayed retirement (see e.g. Stock & Wise 1990, Gruber & Wise 2000, Shoven & Slavov 2014).

- **Accrual Rate (AR) System**: Many countries have PAYG pensions systems in place that are somewhat more sensitive to actual retirement behavior than the DB system. In particular, in these systems the formula pension is reduced if retirement happens before the target age R^*. One popular example of such a system is built on the concept of an "accrual rate". For each period of work the individual earns an accrual rate κ^* that is specified in a way that the system promises the full replacement rate q^* only if the individual retires at the target retirement age $R = R^*$. This means that $\kappa^* = q^* \frac{1}{R^*-A}$ and $\widehat{P}_{\text{AR}}(R, R^*) = \kappa^*(R-A)W = q^* \frac{R-A}{R^*-A} W$.[5]

- **Notional Defined Contribution (NDC) System**: This scheme has been established in Sweden and in a number of other countries and is increasingly popular. Its main principle is that at the moment of retirement at age R the total of contributions that an individual has accumulated over the working life $\tau W(R-A)$ is transformed into a period pension by dividing it by the remaining life expectancy $\omega - R$. This means that $\widehat{P}_{\text{NDC}}(R, R^*) = \tau W \frac{R-A}{\omega - R}$.[6] Therefore in the NDC system the target retirement age R^* does not play any role and the formula pension just reacts to the actual retirement age R.

The pension for early (or late) retirement in each of the three cases $j \in \{\text{DB}, \text{AR}, \text{NDC}\}$ is then given by $P_j = \widehat{P}_j X_j$ (where I skip again the function values). I call the ratio of the final pension P_j to the target pension P_j^* the total pension *reduction*. This reduction might be either due to stipulations of the formula pension $\widehat{P}_j$ or due to the influence of the explicit *deductions* X_j. For the DB system, e.g., the entire reduction follows from the effect of the deductions X_j while for a NDC system the reduction is (primarily) due to the effect of the formula pensions.

The formula pension levels in the defined benefit and the accrual rate system are based on target parameters q^* and κ^*, respectively. It is reasonable to assume that these parameters are fixed in such a fashion that the PAYG system would be balanced in the case when every individual retires at the target retirement age R^* with a target pension P^*. For a constant cohort size N the revenues of the system are in this case given by $I = \tau W(R^* - A)N$ while the expenditures amount to $E = P^*(\omega - R^*)N$. A balanced budget with $E = I$ thus implies $P^* = \tau W \frac{R^*-A}{\omega - R^*}$. For the DB system, this implies a balanced-budget replacement rate of $q^* = \frac{P^*}{W} = \tau \frac{R^*-A}{\omega - R^*}$. Using this relation in

[5] A system like that is, e.g., in place in Austria. The earnings point system in Germany or France can also be directly related to this PAYG variant.

[6] Real-world NDC systems are more complicated due to non-stationary economic and demographic patterns. I will talk about this below in section 3.

Table 1: Three simple PAYG systems

Type (j)	(1) $\widehat{P}_j$	(2) Balanced Target Condition (BTC)	(3) $\widehat{P}_j$ (for BTC)	(4) Ψ_j	(5) $P_j = \widehat{P}_j \widetilde{X}_j$ (for BBC)
DB	$q^* W$	$q^* = \frac{\tau(R^*-A)}{\omega-R^*}$	$\tau W \frac{R^*-A}{\omega-R^*}$	$\frac{\omega-R^*}{\omega-R} \frac{R-A}{R^*-A}$	$\tau W \frac{R-A}{\omega-R} \Delta$
AR	$\kappa^*(R-A)W$	$\kappa^* = \frac{\tau}{\omega-R^*}$	$\tau W \frac{R-A}{\omega-R^*}$	$\frac{\omega-R^*}{\omega-R}$	$\tau W \frac{R-A}{\omega-R} \Delta$
NDC	$\tau W \frac{R-A}{\omega-R}$	—	$\tau W \frac{R-A}{\omega-R}$	1	$\tau W \frac{R-A}{\omega-R} \Delta$

Note: The table shows the formula pension $\widehat{P}_j$, the demographic deduction factor Ψ_j and the total pension $P_j = \widehat{P}_j \widetilde{X}_j$ for three variants of PAYG schemes: DB (Defined Benefit), AR (Accrual Rates), NDC (Notional Defined Contribution). The balanced target condition (BTC) has to hold if the system has a balanced budget in the case that all individuals retire at the target retirement age $R = R^*$. The expression in column (3) follows from inserting column (2) into column (1). The values for Ψ_j in column (4) follow from inserting $\widehat{P}_j$ from column (3) into equation (3) and noting that one can write $\widetilde{X}_j = \Psi_j \Delta$ where $\Delta = 1 + \frac{\delta}{2}(R - R^*) \frac{\omega-A}{R-A}$. Column (5) is the multiple of columns (3), (4) and Δ.

the expressions above we can summarize the formula pension level $\widehat{P}_j$ for the three systems as: $\widehat{P}_{\text{DB}}(R, R^*) = \tau W \frac{R^*-A}{\omega-R^*}$, $\widehat{P}_{\text{AR}}(R, R^*) = \tau W \frac{R-A}{\omega-R^*}$ and $\widehat{P}_{\text{NDC}}(R, R^*) = \tau W \frac{R-A}{\omega-R}$. Note that the balanced budget target pension P^* (at $R = R^*$) is the same in all three systems. For a better overview table 1 contains the expressions that have been derived so far in columns (1) to (3).

One can now insert the pension levels for $\widehat{P}_j$ (column (3) of table 1) into equation (3) in order to derive the (approximated) expressions for the budget-neutral deduction factor $\widetilde{X}_j$ for the three systems $j \in \{\text{DB}, \text{AR}, \text{NDC}\}$. It turns out that this approximated deduction factor can be expressed as: $\widetilde{X}_j = \Psi_j \Delta$ where Ψ_j is a "demographic part" that just depends on the demographic and economic variables ω, A, R and R^*, while Δ is a "financing" part that also depends on the discount rate δ.[7] In particular, $\Psi_{\text{DB}} = \frac{\omega-R^*}{\omega-R} \frac{R-A}{R^*-A}$, $\Psi_{\text{AR}} = \frac{\omega-R^*}{\omega-R}$, $\Psi_{\text{NDC}} = 1$ and $\Delta = 1 + \frac{\delta}{2}(R - R^*) \frac{\omega-A}{R-A}$. These results are collected in column (4) of table 1. Column (5) shows that the application of the deduction factor $\widetilde{X}_j$ leads to an identical final pension payment P_j for all three systems.

[7] One could again write the two coefficients as functions of the various variables, i.e. $\Psi_j = \Psi_j(R, R^*, \omega, A)$ and $\Delta = \Delta(R, R^*, \omega, A, \delta)$. For better readability I again leave out the function arguments.

2.4 Deductions for different discount rates

One can now look at the deductions for various assumptions of the discount rate. At the moment I am not concerned about the budgetary implications of this choice and I am just focusing on the level of deductions that follow from the exact X_j (see equation (1)) or the approximated $\widetilde{X}_j$ (see (3)). In the literature one can find two benchmark assumptions concerning the discount rate which will be discussed below. As a first possibility it is assumed that $\delta = r$, i.e. the discount rate is set equal to the market interest rate. As a second possibility it is argued that the social discount rate should be set to the internal rate of return of a PAYG pension system. In the simple example of this section without economic or population growth the internal rate of return is zero and thus $\delta = 0$.

The latter assumption is a natural starting point. For $\delta = 0$ one gets that $\Delta = 1$ and also $X_{NDC} = \widetilde{X}_{NDC} = 1$. The basic formula of the NDC system $P_{\text{NDC}} = \widehat{P}_{\text{NDC}} = \tau W \frac{R-A}{\omega-R}$ is thus enough to implement the required reduction for early retirement that fullfills the neutrality condition (1). This is different for the two other variants where the pension formula does not suffice to stipulate the necessary reductions even though $\Delta = 1$. In particular, the additional deduction has to be such that the final pension is exactly equal to $P_{\text{NDC}} = \tau W \frac{R-A}{\omega-R}$. For the case of the accrual rate system this means that $\widetilde{X}_{\text{AR}} = \frac{\omega-R^*}{\omega-R}$ while for the DB system one gets that $\widetilde{X}_{\text{DB}} = \frac{\omega-R^*}{\omega-R}\frac{R-A}{R^*-A}$.

For a positive discount rate $\delta > 0$, however, even a NDC system will not lead to long-run stabilization. It is useful to illustrate the magnitude of these effects for realistic numerical values. In particular, we assume that people start to work at the age of $A = 20$, that they die at the age of $\omega = 80$, that the contribution rate is $\tau = 0.25$, the target retirement age $R^* = 65$ and the constant wage $W = 100$. In tables 2 and 3 I show the magnitude of the necessary budget-neutral deductions for the case of $R = 64$ ($R = 60$) and three values of the discount rate δ (0%, 2% and 5%).[8] In order to transform the total deduction factor X into an annual (or rather period) deduction rate x there exist two possibilities. As one possibility one can use the continuous-time framework to conclude from $X = e^{x(R^*-R)}$ that $x = \frac{\ln(X)}{R^*-R}$. In existing pension systems, however, the period deductions are typically expressed in a linear way, i.e. $x = \frac{X-1}{R^*-R}$. In the following tables I show the period deduction rates (in %) based on this linear formula.

All three systems promise a pension of $\widehat{P}(R^*, R^*) = 75$ for a retirement at age $R^* = 65$. For early retirement at $R = 64$ the formula pension is reduced to $\widehat{P}_{\text{NDC}}(R, R^*) = 68.75$

[8] The numbers show X_j, i.e. the exact solutions to equation (1) and not $\widetilde{X}_j$ of the approximated formula (3). The quantitative differences between these two variables are, however, small.

Table 2: Deductions for $R = 64$ and $R^* = 65$

		$\delta = 0$			$\delta = 0.02$			$\delta = 0.05$		
Type j	$\widehat{P}_j$	X_j	x_j(in%)	P_j	X_j	x_j(in%)	P_j	X_j	x_j(in%)	P_j
DB	75.00	0.92	-8.33	68.75	0.90	-9.64	67.77	0.88	-11.81	66.14
AR	73.33	0.94	-6.25	68.75	0.92	-7.59	67.77	0.90	-9.80	66.14
NDC	68.75	1	0	68.75	0.99	-1.43	67.77	0.96	-3.79	66.14

Note: The table shows the actuarial deduction factors X_j, the annual deductions rates x_j (based on the linear relation $x_j = \frac{X_j - 1}{R^* - R}$) and the final pension $P_j(R, R^*) = \widehat{P}_j(R, R^*)X_j$ for three pension schemes and three discount rates. The numerical values are: $A = 20$, $\omega = 80$, $\tau = 0.25$, $W = 100$, $R^* = 65$ and $R = 64$. All cohort members are assumed to reach the maximum age (rectangular survivorship).

Table 3: Deductions for $R = 60$ and $R^* = 65$

		$\delta = 0$			$\delta = 0.02$			$\delta = 0.05$		
Type j	$\widehat{P}_j$	X_j	x_j(in%)	P_j	X_j	x_j(in%)	P_j	X_j	x_j(in%)	P_j
DB	75.00	0.67	-6.67	50.	0.62	-7.70	46.13	0.53	-9.33	40.01
AR	66.67	0.75	-5.00	50.	0.69	-6.16	46.13	0.60	-8.00	40.01
NDC	50.00	1.	0.	50.	0.92	-1.55	46.13	0.80	-4.00	40.01

Note: The table shows the actuarial deduction factors X_j, the annual deductions rates x_j (based on the linear relation $x_j = \frac{X_j - 1}{R^* - R}$) and the final pension $P_j(R, R^*) = \widehat{P}_j(R, R^*)X_j$ for three pension schemes and three discount rates. The numerical values are: $A = 20$, $\omega = 80$, $\tau = 0.25$, $W = 100$, $R^* = 65$ and $R = 60$. All cohort members are assumed to reach the maximum age (rectangular survivorship).

for the NDC system which is the actuarial amount as long as $\delta = 0$. For the accrual rate system, on the other hand, the formula pension is only reduced to $\widehat{P}_{\text{AR}}(R, R^*) = 73.33$ and the system thus needs additional deductions in order to guarantee stability. For the current example the necessary annual deduction rate is 6.25%. For the traditional DB system the annual deduction rate is even larger (8.33%) since there is no adjustment of the pension $\widehat{P}_{\text{DB}}(R, R^*)$. For discount rates above 0 also the NDC needs extra deductions. For $\delta = 0.02$, e.g., the annual deductions are 1.43% and for $\delta = 0.05$ they are 3.79%. For the DB and the AR system the annual deductions also increase by an amount that is somewhat smaller than the extent of δ. If one looks at the even earlier retirement at age $R = 60$ (see table 3) then the results are qualitatively similar. Now the NDC pension is only 50 instead of 75 (for $\delta = 0$) and for the other two systems the annual deductions are somewhat smaller than before.[9]

Summing up, we can conclude that the levels of actuarial deductions depend both on the exact pension formula and on the choice of the social discount rate. For $\delta = 0$ the basic formula of the NDC system is sufficient and no additional deductions are necessary. For the DB and AR systems, however, even for $\delta = 0$ one needs deductions that depend on the demographic structure and on the rules of the pension system. These "demographic deduction factors" are sizable (for our numerical examples between 5% and 8%) and typically larger than the additional deductions that are necessary if one chooses a positive discount rate. In section 3 I will show that these conclusions are also valid in a more general framework.

2.5 Budget-neutral deductions

In the previous sections I have discussed the rates of deduction for different values of the discount rate without looking at the budgetary implications of the various choices. In this section I focus on the appropriate choice to implement a PAYG system that runs a balanced budget. "Budgetary neutrality" requires that retirement before and after the target retirement age does not have an effect on the budget of the pension system in the long run. This definition has two implications. First, one has to consider the *collective* retirement behavior at a specific moment in time in order to evaluate the

[9]The main reason for this is that the deductions now have more time to take force and therefore the annual deductions can be smaller. For the same reasons it also holds that supplements for late retirement (e.g. $R = 66$, results not shown) are larger than the corresponding deductions for early retirement (e.g. $R = 64$). There is less time to reap the benefits of later retirement and therefore the period supplements have to be higher.

budgetary consequences. The look at a single retirement decision is not sufficient since early retirement of an individual might not necessarily cause a budget gap since it could as well be counterbalanced by late retirement of others. Second, the budgetary implications of collective retirement decisions only evolve over time and it is thus necessary to make assumptions about the future development of retirement behavior. As a benchmark for this assessment I will assume that the system has a target density of retirement ages $f^*(R)$. The actual density observed at a specific time is then compared to the target density and the financing needs vis-à-vis this benchmark are the basis for the deductions. I will be more precise on this in section 4. In this section I use some simple examples to emphasize the important point that there are no unique budget-neutral deductions.

2.5.1 A one-time shock in retirement ages

I start with the situation that is dominant in the related literature on actuarial deductions (Börsch-Supan 2004, Werding 2007, Gasche 2012). In particular, the situation is based on the thought experiment that everybody retires at the target retirement age R^* except one individual i who is choosing a different retirement age $R^i = R^L < R^*$. Put differently, there is a one-time shock to retirement and the deductions have to be set in a way as to balance the budget. In this situation the system has to take out a loan at the interest rate $r > 0$ in order to deal with the financial consequences of the early retirement of individual i. In particular, for the time when i is between R^L and R^* years old the revenues are reduced by the missing contributions τW while the expenditures are increased by the pension $P^i_j = \hat{P}^i_j X^i_j$. These extra costs should be borne by the responsible individual (as could, e.g., be argued by invoking the "causality principle"). The right choice of the discount rate is thus given by the market interest rate and the appropriate actuarial deductions X^i_j (including the interest costs) can be calculated from equation (1) by setting $\delta = r$.[10]

2.5.2 A stationary distribution of retirement ages

The one-time-shock scenario is, however, not the only possible constellation of collective retirement choices that involves early retirement. In fact, it is not even a natural benchmark since it refers to a non-stationary situation. In the year after individual i entered retirement there will no longer be a situation where all individuals have an identical re-

[10] This is, e.g., the argument used by Werding (2007) to justify the use of market interest rates to calculate budget-neutral deductions.

tirement age. A stationary distribution would correspond to a situation where in each period of time a constant fraction of retirees chooses the lower retirement age $R^L < R^*$. For this case of stationarity it can be shown (see section 3) that the budget of the PAYG pension system is always balanced without the use of additional deductions if the pension system is based on the NDC type with.

This can be illustrated for a simple stationary distribution where retirement ages can take only two values: a share ϕ of individuals choose an early retirement age $R^L < R^*$ and a share $(1-\phi)$ choose a higher age $R^H > R^L$. The average retirement age is given by $\overline{R} = \phi R^L + (1-\phi)R^H$ which can be different from the target retirement age R^*. The deficit of the PAYG system is given by total expenditures E minus total revenues I which are time-independent and given by:

$$I = \tau WN(\phi(R^L - A) + (1-\phi)(R^H - A)) = \tau WN(\overline{R} - A), \tag{4}$$

$$E = \phi N(\omega - R^L)P_j^L + (1-\phi)N(\omega - R^H)P_j^H. \tag{5}$$

For a NDC system the formula pensions P_j^i are given by $P_{NDC}^L = \tau W \frac{R^L - A}{\omega - R^L}$ and $P_{NDC}^H = \tau W \frac{R^H - A}{\omega - R^H}$, respectively. This implies that total expenditures come out as $E = \tau WN(\overline{R} - A) = I$, i.e. the system is balanced for the stationary two-point distribution of retirement ages. There is no need for loans to finance the early retirement of the L-types, the capital market interest rate is irrelevant, there is no need for extra deductions ($X_{NDC}^L = X_{NDC}^H = 1$) and the appropriate choice of the discount rate is $\delta = 0$.

The intuition behind this result is straightforward. The system needs money to finance the pension of the early retirees with R^L. This is available, however, since in the previous periods the early retirees did not get the full pension for a retirement at age R^* (i.e. $\tau W \frac{R^* - A}{\omega - R^*}$) but rather the smaller pension $P_{NDC}^L = \tau W \frac{R^L - A}{\omega - R^L}$. A similar argument holds for the late retirees where their higher pension can be financed by the extra contributions of the late retirees of future generations.[11]

2.5.3 Additional deductions for a stationary distribution

Even though a discount rate $\delta = 0$ is sufficient for a stable budget in the case of stationarity it is nevertheless interesting to look at a situation where despite this fact the system chooses a positive discount rate $\delta > 0$. In particular, for the two-point distribution of

[11] It is straightforward to transform the example based on the NDC system to the other pension systems by using the demographic deduction factors Ψ_{DB} and Ψ_{AR} from table 1.

section 2.5.2 the general formula for the NDC-pension given in column (5) of table 1 can be written as:

$$P^i_{NDC} = \widehat{P}^i_{NDC}\widetilde{X}^i_{NDC} = \tau W \frac{R^i - A}{\omega - R^i}\left(1 + \frac{\delta}{2}\left(R^i - R^*\right)\frac{\omega - A}{R^i - A}\right), \tag{6}$$

where $i \in \{L, H\}$. Using the expressions for pension revenues (4) and expenditures (5) together with the expressions for P^L_{NDC} and P^H_{NDC} from (6) one can calculate the period deficit $D \equiv E - I$ as:

$$\begin{aligned} D &= \tau W N(\omega - A)\frac{\delta}{2}\left(\phi R^L + (1-\phi)R^H - R^*\right) \\ &= \tau W N(\omega - A)\frac{\delta}{2}\left(\overline{R} - R^*\right) = I\frac{\omega - A}{\overline{R} - A}\frac{\delta}{2}\left(\overline{R} - R^*\right). \end{aligned} \tag{7}$$

Equation (7) confirms the results from above that the budget of the system is in permanent balance if one chooses a discount rate of $\delta = 0$. For the case of a positive discount rate $\delta > 0$ one has to distinguish between two cases. If the target retirement age R^* is set equal to the average actual retirement age $\overline{R}$ then the budget of the system is still in balance, while for $R^* \neq \overline{R}$ this is no longer true. The implications of these two cases are studied in the following.

Balanced budget with higher deductions Equation (7) implies that a positive discount rate is still compatible with a permanently balanced budget as long as $R^* = \overline{R}$. The positive discount rate is, however, not necessary for budgetary reasons since the balance would materialize for *any* value of δ. This means that the higher pensions for the later retirees are paid by the deduction of the early retirees. Whether this is a reasonable and fair property depends on the preferences and the objectives of the system. If, e.g., the main reason for early retirement is seen as a preference for leisure and if (for whatever reason) the policy-makers strive for a higher average retirement age it looks defendable to implement such an unnecessary schema of intragenerational redistribution. If, on the other hand, early retirement is due to bad health or harsh working conditions it might be deemed unfair that those individuals get an additional punishment for their early retirement by paying for the augmented pensions of their more fortunate peers.

Permanent deficits or surpluses with higher deductions An even more severe deviation from the benchmark case with $\delta = 0$ occurs if the discount rate is positive *and* the target retirement age is set above the average retirement age ($R^* > \overline{R}$). In this case equation (7) indicates that the budget will show a permanent surplus ($D > 0$) while there will be a permanent deficit ($D < 0$) in the reverse case ($R^* < \overline{R}$). It is instructive to illustrate this with a numerical example. In particular, assume that $R^L = 62$, $R^H = 67$, $\phi = 0.8$, $R^* = 65$ and again $\omega = 80$ and $A = 20$. It follows that $\overline{R} = 63$. If the social planner chooses a discount rate of $\delta = 0.05$ then the deficit ratio $d \equiv D/I$ comes out as $d = -0.07$. This means that every period the expenditures are 7% lower than the revenues and the system is permanently in surplus.

It is not straightforward to decide whether a situation with a permanent surplus or a permanent deficit is reasonable. In order to do so one would need to specify the optimal size of the PAYG pension system which depends—inter alia—on individual and social preferences, on the economic environment and also on the history of the system. Such an analysis is beyond the scope of the present paper. In fact, in the framework of the simple model presented so far there exists no prima facie reason for a PAYG system since a funded pension system with $r > g$ (where in the simple model $g = 0$) will provide a higher rate of return that does not have to be weighted against potentially higher risk (due to the assumption of complete certainty). In reality, however, the rationale behind the introduction of a PAYG system is typically based on concerns about myopic behavior, intergenerational smoothing, poverty prevention and risk-sharing. As a short-cut one could assume that these concerns have been taken into account in the original design of the PAYG pillar and that the optimal (i.e. socially preferred) size of the system corresponds to the actual revenues at the time of its introduction, i.e. to $I(0) = \tau W N(\overline{R} - A)$. In this case, however, it would not make sense to choose a target level $R^* > \overline{R}$ and a deduction $\delta > 0$ since this would lead to a permanent surplus and thus to a shrinkage of the PAYG system as compared to its optimal size. In particular, the pension for $R^* > \overline{R}$ and $\delta > 0$ would be lower for individuals with $R^i < R^*$ than for the benchmark case with $\delta = 0$. The early retirees would thus pay for the partial dismantling of the PAYG system. For illustrative purposes one can look at an extreme case with $\delta = 0.467$ (and the rest of the parameters as specified above). This huge deduction implies that the L-type (with $R^L = 62$) would not get any PAYG pension ($P^L = 0$) while the H-type (with $R^H = 67$) would get a pension that is about 60% higher than in the benchmark case with $\delta = 0$. On the other hand, the expenditures of this specific PAYG system would only be about 35% of the total revenues. In this extreme scenario the L-types would thus have to save privately in order to have some provision for old age. They are thus treated like the transition generations after the abolishment of a PAYG system who also have to contribute twice (once for the public pension system and once for their private savings).

These are not academic reflections in the framework of a stylized model but they form the background of regular and sometimes rather heated debates on the appropriate level of deductions in many countries. In Germany, some authors have argued for considerably higher deduction rates than the current value of 3.6% in order to implement an incentive-compatible scheme (Börsch-Supan 2004, Werding 2007). Others have countered that it is problematic on normative grounds to choose deductions that are higher than required for budget neutrality just in order to achieve general political goals (Ohsmann et al. 2003).

3 General framework—The case of stationarity

In the previous section I have used a simple set-up to establish a number of results concerning deductions for the case of a stationary demographic situation: (i) a NDC system is stable without the use of additional deductions or supplements; (ii) the DB and AR systems are also compatible with balanced budgets if they are augmented by demographic deduction factors that are independent of the market interest rate; (iii) the discount rate that corresponds to these budget-neutral demographic deduction factors is given by the growth rate of wages; (iv) choosing a higher discount rate might still be associated with a balanced budget if the target retirement age is equal to the average actual retirement age.

In this section I show that all of these results continue to hold in a more general framework with realistic mortality patterns, a general stable distribution of retirement ages and a growing economy. The case of a non-stationary environment is discussed in section 4.

3.1 Set-up

3.1.1 Demographic structure

I work with a model in continuous time. In every instant of time t a new cohort is born. The maximum age that a member of cohort t can reach is time-invariant and denoted by ω. $S(a)$ gives the probability that an individual survives to age a. It holds that $S(0) = 1$, $S(\omega) = 0$ and that survivorship declines with age, i.e. $\frac{dS(a)}{da} \leq 0$ for $a \in [0, \omega]$. The mortality hazard rate is given by $\mu(a) \equiv -\frac{dS(a)}{da}\frac{1}{S(a)}$. Therefore:

$$S(a) = e^{\int_0^a -\mu(x)\,\mathrm{d}x}. \tag{8}$$

An interesting benchmark case is given by rectangular survivorship where $S(a) = 1$ for $a \in [0, \omega]$. In this case there are no premature deaths and all members of a cohort reach the maximum age ω. This corresponds to the assumption made in section 2.1.

Remaining life expectancy is given by:[12]

$$e(z) = \int_z^\omega e^{\int_z^a -\mu(x)\,\mathrm{d}x}\,\mathrm{d}a = \frac{\int_z^\omega S(a)\,\mathrm{d}a}{S(z)}. \tag{9}$$

The second equality follows from the fact that $e^{\int_z^a -\mu(x)\,\mathrm{d}x} = e^{\int_0^a -\mu(x)\,\mathrm{d}x} e^{\int_0^z \mu(x)\,\mathrm{d}x} = \frac{S(a)}{S(z)}$ where the last step uses equation (8).

The size of cohort t at age a is given by $N(a,t) = N(0,t)S(a)$, where $N(0,t)$ stands for the initial size of the cohort. In this section I assume constant sizes of birth cohorts, i.e. $N(0,t) = N, \forall t$. The length of the working life (and thus the number of contribution periods) depends on the starting age and the retirement age. For sake of simplicity I assume that all individuals start to work at age A and are continuously employed up to their individual retirement age R. For the latter I assume that the age-specific probability to retire for generation t is given by $f(a,t)$ for $a \in [A, \omega]$ and that the mortality rates are independent from this probability. The cumulative function $F(a,t)$ then gives the percentage of the surviving members of cohort t that are already retired at age a. It holds that $F(A,t) = 0$ and $F(\omega,t) = 1$. For the moment I focus again on a stationary retirement distribution, i.e. $f(a,t) = f(a)$ and $F(a,t) = F(a)$.

The total size of the active population L and the retired population M are constant and given by:

$$L = N \int_A^\omega S(a)\,(1 - F(a))\,\mathrm{d}a, \tag{10}$$

$$M = N \int_A^\omega S(a)F(a)\,\mathrm{d}a. \tag{11}$$

3.1.2 Budget of the pension system

The contribution rate to the PAYG pension system is assumed to be fixed at τ. For the moment I abstract from intragenerational wage differences and seniority profiles and simply assume that in a specific period t each workers earns an identical wage $W(t)$. Wages grow at rate $g(t)$, i.e. $W(t) = W(0)e^{\int_0^t g(s)\,\mathrm{d}s}$.

Each retired member of generation t receives a pension payment $P(R,a,t)$. The size of the pension can depend on the payment period $t+a$, on the individual's age a and also on

[12] See, e.g., Keyfitz & Caswell (2005) or Goldstein (2006).

the time of his or her retirement $R \leq a$. Below I will say more about the determination of the pension payments in different systems. In appendix A I derive the following expression for total expenditures:

$$E(t) = N \int_A^\omega \left(\int_R^\omega P(R, a, t-a) S(a) \, \mathrm{d}a \right) f(R) \, \mathrm{d}R. \tag{12}$$

Total revenues $I(t)$, on the other hand, are given by:

$$I(t) = \tau W(t) L = \tau W(t) N \int_A^\omega S(a) \left(1 - F(a)\right) \, \mathrm{d}a. \tag{13}$$

The total deficit (or surplus) of the pension system is given by $D(t) = E(t) - I(t)$ while the deficit ratio $d(t)$ is defined as:

$$d(t) = \frac{D(t)}{I(t)} = \frac{E(t)}{I(t)} - 1. \tag{14}$$

A continuously balanced budget is thus characterized by $D(t) = d(t) = 0, \forall t$.

3.2 Different PAYG systems

In appendix B I discuss in detail how the pension level $P_j(R, a, t)$ is determined in the three different pension systems $j \in \{\mathrm{DB}, \mathrm{AR}, \mathrm{NDC}\}$. Here I only summarize the main results. There are two main differences to the simple model of section 2 that have to do with the assumptions of growth and mortality. All pension systems have to specify how pension claims that have been acquired in the past are revalued at the moment of retirement. In the NDC system, e.g., this is done by the choice of a "notional interest rate" $\rho(a, t)$. There exist two popular variants of this interest rate that are discussed in the literature and used in real-world systems:

$$\rho(a, t) = g(t) + \mu(a) \tag{15a}$$

or

$$\rho(a, t) = g(t). \tag{15b}$$

Both notional interest rates reflect the growth rate of average wages $g(t)$ while the first specification (15a) also corrects for the fact that each period some cohort members die. The account values of the deceased cohort members are regarded as "inheritance gains" that are distributed among surviving cohort members by granting an extra return $\mu(a)$.

Furthermore, due to the assumption of a growing economy one also has to specify how pension are adjusted over time. Here it is assumed (for simplicity and in line with the practice in many countries) that ongoing pensions are adjusted with the average growth rate of wages, i.e.

$$\vartheta(t) = g(t). \tag{16}$$

For later reference it is also useful to define the following term:

$$h(R) \equiv \frac{\int_A^R S(a)\,\mathrm{d}a}{(R-A)S(R)}, \tag{17}$$

that stands for the "per capita inheritance gains premium", i.e. the factor by which the first pension at retirement age R is higher if the revaluation takes inheritance gains into account. For rectangular survivorship there are no inheritance gains and thus the average premium is $h(R) = 1$.

In appendix B I show that in this situation the three pension system are associated with the following pension levels:

$$P_{NDC}(R, a, t) = \tau W(t+a)\frac{(R-A)h(R)}{e(R)}X_{NDC}(R, t), \tag{18}$$

$$P_{DB}(R, a, t) = q^* W(t+a) X_{DB}(R, t), \tag{19}$$

$$P_{AR}(R, a, t) = \kappa^* W(t+a)(R-A)X_{AR}(R, t), \tag{20}$$

where the NDC system uses the notional interest including inheritance gains (i.e. equation (15a)) to revalue past contributions. This is in line with the approach used in Sweden. The other two system are based on indexations excluding these mortality adjustments (i.e. on equation (15b)) which is also in line with the real-world systems. All three pension system also allow for the use of demographic adjustment factors $X_j(R, t)$ as discussed in section 2. In the case of non-stationary constellations (see section 4) the deduction factor might also be time-dependent as is indicated by the use of a time index t.

3.3 Budget-neutral deductions

In this part I investigate how the deduction factors $X_j(R, t)$ have to be determined in order to guarantee a balanced PAYG system in the case of a stationary demographic situation. The following proposition states that a standard NDC system with $\rho(a, t) = g(t) + \mu(a)$

leads to a balanced budget without the need for further deductions.

Proposition 1

Assume a stationary demographic situation where the size of birth cohorts is constant ($N(0,t) = N$), people start to work at age A, the maximum age is ω, mortality is described by the survivorship function $S(a)$ for $a \in [0,\omega]$, retirement age is distributed according to the density function $f(R)$ for $R \in [A,\omega]$ and wages grow with rate $g(t)$. In this case a NDC system will be in continuous balance ($E(t) = I(t), \forall t$) if the notional interest rate and the adjustment factor are set according to $\rho(a,t) = g(t) + \mu(a)$ (equation (15a)) and $\vartheta(t) = g(t)$ (equation (16)), respectively, and where there are no additional deductions ($X_{NDC}(R,t) = 1$).

Proof: see appendix C.

Proposition 1 confirms the findings of section 2 in a more general setting. For a stationary economic and demographic situation a NDC system that includes a correction for inheritance gains is stable if one uses the benchmark NDC formula:

$$P_{NDC}(R,a,t) = \tau W(t+a)\frac{(R-A)h(R)}{e(R)}. \tag{21}$$

There is no need for an additional adjustment factor and it holds (as in section 2.5.2) that $X_{NDC}(R,t) = \Psi_{NDC}(R) = 1$.

In this case it is also possible to transform the DB and the AR system into stable systems by just using demographic deduction factors $\Psi_{DB}(R)$ and $\Psi_{AR}(R)$ that are independent of the discount rate δ and of time t. These are calculated in appendix B.[13] As in section 2 I again invoke the "balanced target condition", i.e. the condition that the target replacement rate q^* is chosen in such a way that if everybody retires at the target retirement age R^* there will be no deductions ($\Psi_{DB}(R^*) = 1$) and the system will be in balance. The demographic adjustment factors come out as:

$$\Psi_{DB}(R) = \frac{e(R^*)}{e(R)}\frac{h(R)}{h(R^*)}\frac{R-A}{R^*-A}, \tag{22}$$

$$\Psi_{AR}(R) = \frac{e(R^*)}{e(R)}\frac{h(R)}{h(R^*)}. \tag{23}$$

[13] I also discuss the case of a NDC system that uses (15b) instead of (15a), i.e. that excludes the correction for inheritance gains in the notional interest rate. In this case one also needs a demographic adjustment factor in order to implement a balanced system.

Table 4: Three PAYG systems for a stationary demography

	(1)	(2)	(3)	(4)
Type (j)	$\widehat{P}_j(R,a,t)$	$\widehat{P}_j(R,a,t)$ (for BTC)	Ψ_j	$P_j(R,a,t)$ (for BTC)
DB	$q^*W(t+a)$	$\tau W(t+a)\frac{(R^*-A)h(R^*)}{e(R^*)}$	$\frac{e(R^*)}{e(R)}\frac{h(R)}{h(R^*)}\frac{R-A}{R^*-A}$	$\tau W(t+a)\frac{(R-A)h(R)}{e(R)}$
AR	$\kappa^*W(t+a)$	$\tau W(t+a)\frac{(R-A)h(R^*)}{e(R^*)}$	$\frac{e(R^*)}{e(R)}\frac{h(R)}{h(R^*)}$	$\tau W(t+a)\frac{(R-A)h(R)}{e(R)}$
NDC	$\tau W(t+a)\frac{(R-A)h(R)}{e(R)}$	$\tau W(t+a)\frac{(R-A)h(R)}{e(R)}$	1	$\tau W(t+a)\frac{(R-A)h(R)}{e(R)}$

Note: The table shows the formula pension $\widehat{P}_j(R,a,t)$, the demographic deduction factor Ψ_j and the total pension $P_j(R,a,t) = \widehat{P}_j(R,a,t)\Psi_j$ for three variants of PAYG schemes: DB (Defined Benefit), AR (Accrual Rates), NDC (Notional Defined Contribution). The balanced target condition (BTC) has to hold if the system has a balanced budget in the case that all individuals retire at the target retirement age $R = R^*$. These are specified in the text. Column (4) is the multiple of columns (2) and (3).

In table 4 I collect important formulas for the three PAYG systems. In particular, it contains the formula pension $\widehat{P}_j(R,a,t)$ (both in its basic form and after invoking the balanced budget condition), the demographic deduction factor Ψ_j and the final pension $P_j(R,a,t) = \widehat{P}_j(R,a,t)\Psi_j$. Note that for rectangular survivorship it holds that $h(R) = 1$ and $e(R) = \omega - R$. In this case the results of table 4 coincide with the ones of table 1. In particular, $\Psi_{DB}(R) = \frac{\omega-R^*}{\omega-R}\frac{R-A}{R^*-A}$ and $\Psi_{AR}(R) = \frac{\omega-R^*}{\omega-R}$.

3.4 The choice of discount rates

So far I have shown that for a stationary economic and demographic situation a standard NDC system implements a stable PAYG pension system. By using the correct demographic adjustment factors Ψ_{DB} and Ψ_{AR} also the DB and a AR systems can be amended to guarantee a continuous budgetary balance. This implies that it is not necessary to refer to the market interest rate in order to design the budget-neutral deduction rates in this stationary constellation.

It is interesting to look at this issue from the viewpoint of the standard deduction framework presented in section 2.2 and ask a number of questions. First, which choice of discount rates will give rise to the budget-neutral demographic deduction factors Ψ_j? Second, what deductions are implied if the discount rates are set to higher levels? Third, under which conditions will these higher discount rates also be compatible with a balanced budget? Answers to these questions will be provided in the next three subsections.

3.4.1 The appropriate budget-neutral discount rate for a stationary demography

In order to find the discount rate that is compatible with the budget-neutral demographic deduction factors Ψ_j one has to adapt the neutrality condition (1) of section 2.2 for the general framework. It comes out as:

$$\int_R^{R^*} \left(\tau W(t+a) + \widehat{P}_j(R,a,t)X_j\right) e^{-\delta(a-R)} S(a)\,\mathrm{d}a = \\ \int_{R^*}^{\omega} \left(\widehat{P}_j(R^*,a,t) - \widehat{P}_j(R,a,t)X_j\right) e^{-\delta(a-R)} S(a)\,\mathrm{d}a. \tag{24}$$

I want to know for which choice of δ the total deduction factor will collapse to the demographic factor, i.e. for which δ it holds that $X_j = \Psi_j$. In general, one cannot solve (24) for X_j in closed form. In appendix D I show, however, analytically that for the case of constant growth (i.e. $g(t) = g$) the choice of $\delta = g$ leads to the result that $X_j = \Psi_j$. This means that the appropriate budget-neutral discount rate for a stationary situation is given by the internal growth rate of wages. This has often been claimed in the related literature but the present framework allows to formulate it in a precise manner and to state the exact conditions (in particular demographic stationarity) under which it holds.

3.4.2 Deductions for different discount rates

In a next step one can investigate which deductions are implied by choices of the discount rate $\delta > g$. Although these choices are not necessary from the viewpoint of budgetary stability it is nevertheless instructive to see the magnitudes involved. To do so I use illustrative numerical examples. In particular, I assume a Gompertz survival curve of the form $S(a) = e^{\frac{\alpha}{\beta}\left(1-e^{\beta a}\right)}$ which has a mortality rate of $\mu(a) = \alpha e^{\beta a}$ (i.e. the logarithm of mortality rates increases linearly in age).[14]

In Tables 5 and 6 I report the deduction factors X_j and annual deduction rates x_j for various assumption concerning the discount rate $\hat{\delta} \equiv \delta - g$. They correspond to tables 2 and 3 from section 2.2 which were based on rectangular survivorship. In addition, I also report the demographic deduction factors Ψ_j that are sufficient for budgetary stability. For all three systems the target pension level at the target retirement age $R^* = 65$ is given

[14]The Gompertz-function delivers a good description of empirical mortality data. For the following examples I use a parameterization of $\alpha = 0.000025$ and $\beta = 0.096$. This is roughly based on the Austrian life tables from 2005 which have the convenient property that the (unisex) life expectancy at birth was almost exactly 80 which facilitates comparisons to the numerical examples of section 2 with rectangular survivorship.

Table 5: Deductions for $R = 64$ and $R^* = 65$

			$\hat{\delta}=0$			$\hat{\delta}=0.02$			$\hat{\delta}=0.05$		
Type j	$\widehat{P}_j$	Ψ_j	X_j	x_j(in%)	P_j	X_j	x_j(in%)	P_j	X_j	x_j(in%)	P_j
DB	67.0	0.93	0.93	-7.04	62.3	0.91	-8.71	61.1	0.89	-11.49	59.3
AR	65.5	0.95	0.95	-4.92	62.3	0.93	-6.64	61.1	0.91	-9.48	59.3
NDC	62.3	1	1	0	62.3	0.98	-1.81	61.1	0.95	-4.79	59.3

Note: The table shows the actuarial deduction factors X_j, the annual deductions rates x_j (based on the linear relation $x_j = \frac{X_j - 1}{R^* - R}$) and the final pension $P_j(R, R^*) = \widehat{P}_j(R, R^*)X_j$ for three pension schemes and three discount rates $\hat{\delta} \equiv \delta - g$. For the sake of comparison also the values of the pure demographic deduction factors Ψ_j are reported. The numerical values are: $A = 20$, $\tau = 0.25$, $g = 0.02$, $R^* = 65$ and $R = 64$. For all three schemes the target pension is $P^* = 67$. Mortality follows a Gompertz distribution with $\alpha = 0.000025$ and $\beta = 0.096$.

Table 6: Deductions for $R = 60$ and $R^* = 65$

			$\hat{\delta}=0$			$\hat{\delta}=0.02$			$\hat{\delta}=0.05$		
Type j	$\widehat{P}_j$	Ψ_j	X_j	x_j(in%)	P_j	X_j	x_j(in%)	P_j	X_j	x_j(in%)	P_j
DB	67.0	0.7	0.7	-5.94	47.1	0.64	-7.28	42.6	0.53	-9.36	35.6
AR	59.5	0.79	0.79	-4.18	47.1	0.72	-5.69	42.6	0.6	-8.04	35.6
NDC	47.1	1	1	0	47.1	0.9	-1.91	42.6	0.76	-4.87	35.6

Note: See table 5 with the difference that now $R = 60$ instead of $R = 64$.

by $P^* = 67$ (which implies a target replacement rate for the DB system of $q^* = 0.67$). This is lower than for the case of rectangular survivorship since —due to premature deaths— the difference in remaining life expectancy $e(R^*)$ at $R^* = 65$ is now larger (18.6 vs. 15). The step-up of pensions due to inheritance gains is non-trivial and given by $h(R^*) = 1.11$ (or 11%). This is due to the fact that only 88% of all members of a cohort survive up to this age ($S(R^*) = 0.88$).

The results are qualitatively similar to the ones for rectangular survivorship in tables 2 and 3. The first thing to note is that for $\hat{\delta} = 0$ (i.e. $\delta = g$) the total deductions correspond exactly to the demographic adjustment factors, i.e. $X_j = \Psi_j$ as has already been shown in section 3.4.1. For the DB system the correct annual deduction rate for a retirement at the age of 64 is 7% while it is 4.9% for the AR system if mortality follows a Gompertz pattern. This is smaller than the corresponding rates for rectangular survival where they have been calculated as 8.33% and 6.25%, respectively. For larger discount rates, however, the difference shrinks and for $\hat{\delta} = 0.05$, e.g., the annual deduction rates for the Gompertz and the rectangular case are almost identical (11.49%, 9.48% and 4.79% vs. 11.81%, 9.8% and 3.79%). A similar conclusion holds for the case of $R = 60$ where the differences between the results of table 6 (Gompertz) and table 3 (rectangular) are even smaller.

3.4.3 Balanced and unbalanced budgets for different discount rates

In the simple framework of section 2 I have shown that even for discount rates that are larger than necessary it might still be the case that the system runs a balanced budget. In particular, I have shown in section 2.5.3 that the balanced budget condition depends on the choice of both the discount rate δ and the target retirement age R^*. This can be repeated for the general framework. Again it is not possible to derive closed-form solutions for the balanced budget and one has to resort to numerical calculations as reported in appendix E where I use various assumptions concerning the density distribution of retirement ages $f(R)$. These density functions all stem from the family of triangular distributions but they differ in the mean and in the mode retirement age.

The results are completely parallel to the ones of section 2.5. In particular, the budget is balanced as long as the discount rate is set equal to the growth rate of wages (as has already been shown analytically in section 3.4.1). Furthermore, the budget also turns out to be (approximately) balanced for situations where the discount rate differs from this benchmark value as long as $R^* = \overline{R}$.

4 General framework—The case of non-stationarity

In section 2.5 I have argued why it is necessary to make assumptions about the collective retirement behavior of past, current and future retirees in order to determine the level of budget-neutral deductions. For the case of stationarity (as discussed in section 3) this has been straightforward since the collective behavior is completely described by the stationary retirement distribution $f(R)$. For the case of non-stationarity, however, this is different and one needs some reference scenario against which one can judge the retirement behavior of an individual or a generation.

A first possibility is to assume that the policymaker has some stable "normative reference distribution" $f^*(R)$ for the retirement behavior of the individuals that retire in each period. One could think of this, e.g., as the actual distribution at the introduction of the PAYG system or after a large reform. Alternatively, the reference distribution could of course also represent the target of the policymaker that he would like the retirees to follow. This reference retirement pattern can then be compared to the actual retirement density $f(R, t-R)$ in this period.[15] If the actual density deviates from the reference density then the pension system will introduce a period-specific deduction factor $X(t)$ that should be sufficient to cover all extra expenses (including interest rate costs) that are due to this "collectively deviant behavior" of the retiring cohort.

This can be translated into a formal language by using the concept of the social security wealth (see section 2.2, equation (2)). The variable $SSW(A, R, X(t), \delta)$ measures the net-benefits (or net costs) of a PAYG system with the pension rule $\widehat{P}$ and the deduction $X(t)$ for an individual who retires at the age R, with a discount rate δ and at the moment of labor market entry A. I focus on a simple example with rectangular survivorship ($S(a) = 1$), zero growth ($g = 0$) and a NDC system. The SSW also reflects the costs of the pension rules to the system itself (possibly for a different discount rate δ). One can use this concept to calculate the total costs (or benefits) to the system that are due to the fact that the retirees in period t followed the pattern $f(R, t-R)$ instead of the reference pattern $f^*(R)$. The actual retirement pattern is associated with a total pension net wealth $TW(t)$ of:

$$TW(t) = \int_A^\omega SSW(A, R, X(t), r) f(R, t-R)\, \mathrm{d}R, \tag{25}$$

where the market interest rate r is used as the discount rate. For the reference scenario this total net wealth is given by:

$$TW^*(t) = \int_A^\omega SSW(A, R, 1, r) f^*(R)\, \mathrm{d}R, \tag{26}$$

where I use the fact that for a stationary distribution (as is the case for $f^*(R)$) the basic rules of the NDC system do not need a deduction factor (i.e. $X^*(t) = 1$).[16] The deduction factor $X(t)$ for all individuals that retire in period t can then be calculated

[15] Note that $f(R, s)$ denotes the retirement density of the cohort born in period s. In period t the mass of individuals retiring at age R is therefore given by $f(R, t-R)$.

[16] Note that for a stationary distribution I have shown in section 3 that the individual SSW and the budget of the system is balanced for discounting with the implicit rate of return and without an additional deduction factor, i.e. $SSW(A, R, 1, g) = 0$. The calculation of (26), however, uses discounting with $\delta = r$ since $TW^*(t)$ measures the the present value of the payment streams.

from the condition:

$$TW(t) = TW^*(t). \tag{27}$$

This formulation, however, has the implication that every retiree faces the same deduction factor $X(t)$ independent of his or her retirement age. It might therefore looks more reasonable to assume an *annual* deduction rate $x(t)$ that is identical for every retiree in period t but that leads to individual-specific total deduction factor $X(R,t) = e^{x(t)(R^*-R)}$.

One can use (25), (26), the definition of SSW in (2) and the formula $\widehat{P}_{\text{NDC}} = \tau W \frac{R-A}{\omega-R}$ for the NDC system to express (27) as:

$$\begin{aligned} \int_A^\omega \int_R^\omega \tfrac{R}{\omega-R} X(t) e^{-\delta(a-A)} \,\mathrm{d}a f(R, t-R) \,\mathrm{d}R - \int_A^\omega \int_R^\omega \tfrac{R}{\omega-R} e^{-\delta(a-A)} \,\mathrm{d}a f^*(R) \,\mathrm{d}R = \\ \int_A^\omega \int_A^R e^{-\delta(a-A)} \,\mathrm{d}a \left(f(R, t-R) - f^*(R)\right) \mathrm{d}R \end{aligned} \tag{28}$$

If the actual distribution is also stationary at $f(R)$ then one is back to a case discussed in section 2.5. In principle a stationary distribution is stable without the use of any deductions and with a discount rate of $\delta = g$. This also comes out of (28).

The benchmark in the literature is an extreme version of this comparison to a stable target distribution. In particular, $f^*(R)$ is assumed to be given by the degenerate distribution with the complete mass at $R = R^*$ and zero otherwise. As a starting point it is useful to note that in section 2 I have already discussed one simple case of a non-stationary distribution of retirement ages, namely the case where every member of every cohort retires at the target age R^* while one single individual of some specific cohort retires at an earlier age $R^i < R^*$. In this situation the PAYG system needs to take up a loan in order to finance the extra costs that are due to the early retirement.

Another possibility beside the assumption of a stable target distribution $f^*(R)$ is a rather pragmatic assumption that takes the density function of the existing retirement ages (i.e. the average over $f(R, t-a)$) as the benchmark for the cohort retiring in period t. In this case the target distribution would be itself time-varying and has to be written as $f^*(R, t-R)$. One can show that such a target distribution would lead to an approximately balanced budget in the long-run if there is no trend in retirement behavior. Normally the retirement distribution only changes slowly. We have make some simulations and the resulting deductions are close to the demographic deduction factors (i.e. to a discount rate based on the implicit rate of return g and not the market interest rate r). Detailed results will be included in a future version of the paper.

5 Extensions

In the paper I have focused on simple economic and demographic set-ups in order to derive important properties and results about deduction factors in an intuitive and often analytical manner. Real-world environments and pension rules involve much more elements. As far as pension rules are concerned there are disability and survivor benefits and also minimum pensions. These additional programs change the pattern of payment flows and it is ultimately a political (or normative) question whether minimum pensions, e.g. are viewed as being part of the regular pension system or whether they should be financed by general taxes (and thus be excluded from the considerations about actuarial deductions). As far as the demographic and economic environment is concerned the existing complexities include: different entry ages into the labor market and patchy work histories; changing population sizes, differential fertility and increasing fertility ages; different levels of eduction, average wage sand wage patterns. Finally, income and eduction are typically correlated with demographic variables like fertility and mortality.

I have to leave most of these important factors and questions for future research and only sketch some answers. This part of the paper will be elaborated in a future version. One important observation is that most of the arguments made above hold if one assumes that demographic and economic variables are independently distributed (or uncorrelated).

6 Conclusions

Annual pensions for early retirement have to be lower than pension at the target retirement age. But how much lower in order to keep the budget of a PAYG system in balance? In this paper I have provided an answer to this question that is important for every pension system. I have shown that the answer depends on three crucial issue. First, it depends on the nature of the pension system. The rate of deduction can be lower in systems where the formula pension is already leading to a reduction in pension benefits (like in NDC or AR systems). Second, the level of budget-neutral deduction also depends on the collective retirement behavior over time. In particular, I have shown that a NDC system is stable just by following the pension rule without the need for any further deductions if the retirement distribution is stable. The third important factor for budget-neutral deductions is the choice of the discount factor. In situations with a stable retirement distribution it can be chosen to be equal to the implicit rate of return. This is in contrast to the benchmark scenario in the literature that is based on a one-time shock, i.e. on a constellation where everybody retires at the target age and only one individual at a lower age. In this case it is justified to discount future payment streams with the market interest rate. I have also shown that under certain assumptions (e.g. that the average retirement age is equal to the target age) a higher discount rate than necessary is also compatible with a balanced budget It is not clear, however, whether this is a reasonable approach since early retirees will have to face larger cuts than required for budget neutrality while late retirees will be rewarded with extra benefits.

For future research it would be interesting to use numerical methods to calculate the budget-neutral deductions in larger models that include many of the economic, demographic and policy details of real-world systems. Furthermore, the analysis in this paper has been based on the assumption of exogenous retirement behavior. Retirement behavior, however, is of course also influenced by the pension rules including the rates of deduction. Since the budget-neutral deduction rates depend on current an future retirement behavior while the latter will react to the deductions one might expect interesting interactions that will be studied in future work.

References

Börsch-Supan, A. (2004), Faire Abschläge in der gesetzlichen Rentenversicherung, Sonderforschungsbereich 504 Publications 04-63, Sonderforschungsbereich 504, University of Mannheim.

Coile, C., Diamond, P., Gruber, J. & Jousten, A. (2002), 'Delays in Claiming Social Security Benefits', *Journal of Public Economics* **84**(3), 357–385.

Gasche, M. (2012), ' Was Sind die Richtigen Rentenabschläge? Neue Perspektiven', *Review of Economics* .

Goldstein, J. (2006), 'Found in Translation?: A Cohort Perspective on Tempo-Adjusted Life Expectancy', *Demographic Research* **14**(5), 71–84.

Gruber, J. & Wise, D. (2000), Social Security Programs and Retirement Around the World, *in* 'Research in Labor Economics', Emerald Group Publishing Limited, pp. 1–40.

Gustman, A. L. & Steinmeier, T. L. (2015), 'Effects of social security policies on benefit claiming, retirement and saving', *Journal of Public Economics* **129**(C), 51–62.

Keyfitz, N. & Caswell, H. (2005), *Applied Mathematical Demography*, Springer, New York, ed. 3.

Knell, M. (2016), 'Increasing Life Expectancy and NDC Pension Systems', *Journal of Pension Economics and Finance* **FirstView**, 1–30.

Ohsmann, S., Stolz, U. & Thiede, R. (2003), 'Rentenabschläge bei Vorzeitigem Rentenbeginn: Was ist Versicherungsmathematisch Fair?', *Die Angestellten Versicherung* **4**, 1–9.

Queisser, M. & Whitehouse, E. R. (2006), Neutral or Fair?: Actuarial Concepts and Pension-System Design, OECD Social, Employment and Migration Working Papers 40, OECD, Directorate for Employment, Labour and Social Affairs.

Shoven, J. B. & Slavov, S. N. (2014), 'Does It Pay to Delay Social Security?', *Journal of Pension Economics and Finance* **13**(2), 121144.

Stock, J. H. & Wise, D. A. (1990), 'Pensions, the Option Value of Work, and Retirement', *Econometrica* **58**(5), 1151–1180.

Werding, M. (2007), 'Versicherungsmathematisch Korrekte Rentenabschlge fr die Gesetzliche Rentenversicherung', *Ifo Schnelldienst* **60**(16), 19–32.

Appendices

A Total expenditures (section 2.1)

In order to calculate the total expenditures of the pension system for the general framework in section 3 one can make the following considerations. First, focus on one particular retirement age R and calculate the total of pension payments that is distributed to the group of pensioners that has retired at this age. This comprises individuals at different ages $a \in [R, \omega]$. For a person who is of age a in period t the pension payment is $P(R, a, t-a)$ and the size of this subgroup is $N \times S(a)$. The total payments to people with retirement age R in period t is thus given by: $P^{total}(R,t) = N \int_R^\omega P(R,a,t-a)S(a)\,\mathrm{d}a$. The same logic applies for any possible retirement age $R \in [A, \omega]$ where the relative frequency of the retirement age is given by $f(R)$. Total pension expenditures in period t can thus be written as $E(t) = \int_A^\omega P^{total}(R,t)f(R)\,\mathrm{d}R$ or as the formulation stated in equation (12) of the paper.[17]

B Different pension systems (section 3.2)

B.1 Notional defined contribution pension system

I start with the discussion of how the pension level $P(R, a, t)$ is determined in NDC systems. This provides again a useful benchmark case to derive the necessary deductions and supplements for the two other PAYG pension systems (AR and DB).

The contributions in a NDC system are credited to a notional account and they are revalued with a “notional interest rate” $\rho(a, t)$ (that is allowed to change over time and across ages). The total value of this account is called the “notional capital” that accumulates over the working periods of an insured person. When the individual retires at age R the final notional capital is given by:

$$K(R,t) = \int_A^R \tau W(t+x) e^{\int_x^R \rho(s,t+s)\,\mathrm{d}s}\,\mathrm{d}x, \tag{29}$$

[17] Alternatively, one could also reverse the order and look first at a fixed age a of the retired population and calculate the total pension payments to those individuals that have different retirement ages R. In the second step one would then calculate the total pension payments to all possible ages a. This results in an equivalent expression. I focus on the formulation in (12) since it is more convenient for later calculations.

where the cumulative factor $e^{\int_x^R \rho(s,t+s)\,\mathrm{d}s}$ indicates how the contribution $\tau W(t+x)$ that is paid into the pension system in period $t+x$ is revalued when calculating the final amount of the notional capital in period $t+R$ (the period of retirement). The notional interest rate is a crucial magnitude in a NDC system as I discuss in a different paper (see Knell 2016). In the paper I use two standard definitions that can be found in real-world NDC systems. Both notional interest rates are related to the growth rate of productivity (or of average wages) while one also contains a correction for the fast that the cohort size decreases with the mortality rate $\mu(a)$. The account values of the deceased cohort members are regarded as "inheritance gains" that are distributed among surviving cohort members by granting an extra return $\mu(a)$. In particular:

$$\rho(a,t) = g(t) + \mu(a)$$
or
$$\rho(a,t) = g(t),$$

which corresponds to equations (15a) and (15b) in the paper. Using these definitions for the notional interest rate in (29) one can conclude that $K(R,t) = \tau W(t+R)(R-A)$ if one uses the value of $\rho(a,t)$ without inheritance gains (equation (15b)) or $K(R,t) = \tau W(t+R) \int_A^R e^{\int_a^R \mu(s)\,\mathrm{d}s}\,\mathrm{d}a$ if one uses the specification that includes the inheritance gains (equation (15a)). This can also be written as $K(R,t) = \tau W(t+R)(R-A)h(R)$, where $h(R) \equiv \frac{\int_A^R S(a)\,\mathrm{d}a}{(R-A)S(R)}$ as expressed in equation (17). The term $h(R)$ stands for the per capita "inheritance gains premium" to the "normal" notional capital, averaged over the $(R-A)$ contribution periods and distributed among the mass $S(R)$ of surviving members.[18]

The first pension that is received by a member of cohort t who retires at the age R is given by:

$$P_{NDC}(R,R,t) = \frac{K(R,t)}{e(R)} X_{NDC}(R,t). \tag{31}$$

The first pension is calculated by taking the final notional capital $K(R,t)$ and turning it into an annual pension payment by using remaining life expectancy $e(R)$ as the annuity conversion factor. In addition, there might also be a deduction factor $X_{NDC}(R,t)$ that is applied to secure a balanced budget if the formula pension $\frac{K(R,t)}{e(R)}$ is not sufficient. For simplicity I write this demographic deduction factor only as a function of the actual

[18] The relation follows by noting that $e^{\int_a^R \mu(s)\,\mathrm{d}s}$ can be written as: $e^{\int_a^R \mu(s)\,\mathrm{d}s} = e^{\int_0^R \mu(s)\,\mathrm{d}s} e^{\int_0^a -\mu(s)\,\mathrm{d}s}$. Therefore $\int_A^R e^{\int_a^R \mu(s)\,\mathrm{d}s}\,\mathrm{d}a = e^{\int_0^R \mu(s)\,\mathrm{d}s} \int_A^R e^{\int_0^a -\mu(s)\,\mathrm{d}s}\,\mathrm{d}a = (S(R))^{-1} \int_A^R S(a)\,\mathrm{d}a$, where I use (8).

retirement age R and time although—in general—it will also depend on other demographic and policy variables (like R^* and A). The dependence on time is relevant for the case of non-stationary retirement distributions.

From equation (9) it is known that:

$$e(R) = \frac{\int_R^\omega S(a)\,\mathrm{d}a}{S(R)}. \tag{32}$$

This can be used together with $K(R,t) = \tau W(t+R)(R-A)h(R) = \tau W(t+R)\frac{\int_A^R S(a)\,\mathrm{d}a}{S(R)}$ to derive the first pension in the case that the notional interest rate includes a correction for the inheritance gains (i.e. $\rho(a,t) = g(t) + \mu(a)$):

$$\begin{aligned} P_{NDC}(R,R,t) &= \tau W(t+R)\frac{(R-A)h(R)}{e(R)}X_{NDC}(R,t) \\ &= \tau W(t+R)\frac{\int_A^R S(a)\,\mathrm{d}a}{\int_R^\omega S(a)\,\mathrm{d}a}X_{NDC}(R,t). \end{aligned} \tag{33}$$

This expression is quite intuitive. At the moment of retirement the first pension payment is proportional to the wage level in the period of retirement $t+R$. This is due to the fact that past contributions are indexed to average wage growth. The inclusion of inheritance gains raises the notional capital (which is captured by the expression in the numerator) while the period pension payment depends on remaining life expectancy (which is captured by the expression in the denominator). In addition there might be a correction for early or late retirement $X_{NDC}(R,t)$.

For the situation where inheritance gains are not included and where the notional interest rate is simply given by $\rho(t) = g(t)$ the first pension is:

$$\begin{aligned} P_{NDC'}(R,R,t) &= \tau W(t+R)\frac{R-A}{e(R)}X_{NDC'}(R,t) \\ &= \tau W(t+R)\frac{R-A}{\frac{\int_R^\omega S(a)\,\mathrm{d}a}{S(R)}}X_{NDC'}(R,t). \end{aligned} \tag{34}$$

For the case of rectangular survivorship ($S(a) = 1$ for $a \in [0,\omega]$) one gets that both (33) and (34) lead to the same result that $P_{NDC}(R,R,t) = \tau W(t+R)\frac{R-A}{\omega-R}X_{NDC}(R,t)$. This is the same expression that was used in section 2 (see table 1).

Existing pensions are adjusted according to:

$$P_j(R,a,t) = P_j(R,R,t)e^{\int_R^a \vartheta(t+s)\,\mathrm{d}s}, \tag{35}$$

for $a \in [R, \omega^c(t)]$ and where I use the index j since the adjustment in (35) is valid for all three pension systems $j \in \{\mathrm{DB}, \mathrm{AR}, \mathrm{NDC}\}$. The variable $\vartheta(t)$ stands for the adjustment rate in period t and the cumulative adjustment factor $e^{\int_R^a \vartheta(t+s)\,\mathrm{d}s}$ indicates how the first pension $P(R,R,t)$ is adjusted to give the pension payment in period $t+a$. In the paper I assume that ongoing pensions are adjusted with the average growth rate of wages as expressed in equation (16) stating that $\vartheta(t) = g(t)$.

In this case one can use (33) and (35) to conclude that with $\rho(a,t) = g(t) + \mu(a)$ the ongoing pension is:

$$P_{NDC}(R,a,t) = \tau W(t+a)\frac{(R-A)h(R)}{e(R)}X_{NDC}(R,t),$$

which is equation (18) in the paper. For $\rho(a,t) = g(t)$ it holds that $P_{NDC'}(R,a,t) = \tau W(t+a)\frac{R-A}{e(R)}\Psi_{NDC'}(R)$. In the following I focus on the first case with a compensation for inheritance gains.

B.2 Defined benefit pension system

In a similar vein one can look at the two alternative pension systems discussed in section 2. The defined benefit (DB) system promises a target replacement rate q^* if an individual retires at the target retirement age R^*. The replacement rate is related to average lifetime income, where past incomes are revalued at a rate $\rho(a,t)$ and where there are correction for early/late retirement. In particular, instead of (31) it now holds that:

$$P_{DB}(R,R,t) = q^*\overline{W}^{LT}(R,t)X_{DB}(R,t), \tag{36}$$

where $\overline{W}^{LT}(R,t) = \frac{\int_A^R W(t+x)e^{\int_x^R \rho(s,t+s)\,\mathrm{d}s}\,\mathrm{d}x}{R-A}$. This expression is closely related to the notional capital (29) for NDC systems. I know of no existing DB system that includes a correction for inheritance gains when indexing past wage levels. Therefore the benchmark DB system is characterized by the indexation $\rho(a,t) = g(t)$. From this is follows that $P_{DB}(R,R,t) = q^*W(t+R)X_{DB}(R,t)$. As above I assume that existing pension are adjusted with the growth rate of average wages according to (35) and (16) and thus:

$$P_{DB}(R,a,t) = q^* W(t+a) X_{DB}(R,t),$$

which corresponds to equation (19) in the paper.

B.3 Accrual rate pension system

Finally, one can look at the accrual rate system in the general set-up. The AR system promises a pension that is proportional to the revalued average lifetime income. In particular, for each year of work the system promises a certain percentage κ^* (the accrual rate) of this lifetime average that can be claimed at the target retirement age R^*. For early retirement there exists a deduction $X_{AR}(R,t)$. In particular, the first pension is now defined as:

$$P_{AR}(R,R,t) = \kappa^*(R-A)\overline{W}^{LT}(R,t) X_{AR}(R,t), \tag{37}$$

where $\overline{W}^{LT}(R,t)$ stands for lifetime income (as defined in section B.2) for which past incomes are revalued at a rate $\rho(a,t)$. As before and in line with existing AR system I assume that indexation follows the growth rate of average wages, i.e. $\rho(a,t) = g(t)$. From this is follows that $P_{AR}(R,R,t) = \kappa^* W(t+R)(R-A)X_{AR}(R,t)$. For pension adjustment according to (35) and $\vartheta(t) = g(t)$ one can conclude that:

$$P_{AR}(R,a,t) = \kappa^* W(t+a)(R-A)X_{AR}(R,t).$$

which corresponds to equation (20) in the paper.

B.4 Demographic adjustment factors

Proposition 1 shows that for a stationary retirement distribution $f(R)$ the NDC system is stable for $X(R,t) = \Psi(R) = 1$. It is now straightforward to discuss the demographic deductions $\Psi_j(R)$ that are necessary to establish balanced budget for pension systems that deviate from the NDC benchmark. This can be seen by looking at equation (40) (see below in section C). If the pension payments of the alternative system can be written as $P_j(R,a,t) = \tau W(t+a)(\cdots)$ then the correction $\Psi_j(R)$ just has to be set in a way such that it mimics the benchmark NDC-pension given in (21). As a first example one can look at a NDC' system that does not include the compensation for inheritance gains (as is the case for most existing NDC systems with the notable exception of Sweden) and that sets $\rho(a,t) = g(t)$. In this case it has been shown above that the pension is given by $P_{NDC'}(R,a,t) = \tau W(t+a)\frac{(R-A)}{e(R)}\Psi_{NDC'}(R)$. It is immediately apparent that an adjustment with $\Psi_{NDC'}(R) = h(R)$ leads to a balanced budget. Otherwise, the pension system would run a surplus since $h(R) > 1$, i.e. the system would keep the inheritance gains for itself instead of distributing them among the surviving population.

For the defined benefit system I invoke as in section 2 the "balanced target condition", i.e. I assume that the target replacement rate q^* is associated with a situation that there will not be any deductions ($\Psi_{DB}(R^*) = 1$) if everybody retires at the target retirement age R^*. For the NDC system we know from equation (21) that a balanced budget with $R = R^*, \forall i$ requires that everybody gets a pension equal to $P_{NDC}(R^*,a,t) = \tau W(t+a)\frac{(R^*-A)h(R^*)}{e(R^*)}$. This should be equal to the DB pension with $R = R^*, \forall i$, i.e. to $P_{DB}(R^*,a,t) = q^* W(t+a)$. From these two expressions it follows that $q^* = \tau\frac{(R^*-A)h(R^*)}{e(R^*)}$. Inserting this into equation (19) for $P_{DB}(R,a,t)$ leads to the DB pension after evoking the stability condition:

$$P_{DB}(R,a,t) = \tau W(t+a)\frac{(R^*-A)h(R^*)}{e(R^*)}\Psi_{DB}(R). \tag{38}$$

This expression can now be compared to the pension of the benchmark NDC system (21) (that leads to a balanced budget) to conclude that $\Psi_{DB}(R) = \frac{R-A}{R^*-A}\frac{e(R^*)}{e(R)}\frac{h(R)}{h(R^*)}$ as stated in equation (22).

One can use similar steps for the AR pension system. In particular, I assume that the target accrual rate κ^* is chosen in such a way that the system is balanced if everybody retires at the target retirement age $R = R^*, \forall i$. Inserting the implied target accrual rate $\kappa^* = \tau\frac{h(R^*)}{e(R^*)}$ into equation (20) for $P_{AR}(R,a,t)$ leads to the AR pension after evoking the stability condition:

$$P_{AR}(R,a,t) = \tau W(t+a)\frac{(R-A)h(R^*)}{e(R^*)}\Psi_{AR}(R). \tag{39}$$

The deduction rate that is necessary for a balanced AR system can be calculated by setting (39) equal to the NDC pension (21) and solving for $\Psi_{AR}(R)$. This leads to $\Psi_{AR}(R) = \frac{e(R^*)}{e(R)}\frac{h(R)}{h(R^*)}$ as stated in equation (23). These and other important formulas are collected in table 4 in the paper.

C Proof of proposition 1 (section 3.3)

For the NDC system one can insert the pension level from equation (18), i.e. $P_{NDC}(R,a,t-a) = \tau W(t)\frac{(R-A)h(R)}{e(R)}X_{NDC}(R,t)$ into (12) to conclude that:

$$\begin{aligned} E(t) &= \tau W(t)N\int_A^\omega \frac{(R-A)h(R)}{e(R)}X_{NDC}(R,t)\left(\int_R^\omega S(a)\,\mathrm{d}a\right)f(R)\,\mathrm{d}R \\ &= \tau W(t)N\int_A^\omega\left(\int_A^R S(a)\,\mathrm{d}a\right)X_{NDC}(R,t)f(R)\,\mathrm{d}R, \end{aligned} \tag{40}$$

where I use the definitions $h(R) = \frac{\int_A^R S(a)\,\mathrm{d}a}{(R-A)S(R)}$ and $e(R) = \frac{\int_R^\omega S(a)\,\mathrm{d}a}{S(R)}$.

For the assumptions of proposition 1 total expenditures in equation (40) can be written as:

$$E(t) = \tau W(t)N\int_A^\omega\left(\int_A^R S(a)\,\mathrm{d}a\right)f(R)\,\mathrm{d}R.$$

One can define $u(R) = \int_A^R S(a)\,\mathrm{d}a$ and $v(R) = 1 - F(R)$ with $u'(R) = S(R)$ and $v'(R) = -f(R)$. Using integration by parts it holds that:

$$\int_A^\omega\left(\int_A^R S(a)\,\mathrm{d}a\right)f(R)\,\mathrm{d}R = -\int_A^\omega u(R)v'(R)\,\mathrm{d}R = -u(R)v(R) + \int_A^\omega u'(R)v(R)\,\mathrm{d}R.$$

The term $(-u(R)v(R))$ is given by $\left[\left(\int_A^R S(a)\,\mathrm{d}a\right)(1-F(R))\right]_A^\omega$ which can be evaluated as

$$\left(\int_A^\omega S(a)\,\mathrm{d}a\right)(1-F(\omega)) - \left(\int_A^A S(a)\,\mathrm{d}a\right)(1-F(A)) = 0. \tag{41}$$

Since it holds that $\int_A^\omega u(R)v'(R)\,\mathrm{d}R = \int_A^\omega S(R)(1-F(R))\,\mathrm{d}R$ one can conclude that:

$$E(t) = \tau W(t)N\int_A^\omega S(R)(1-F(R))\,\mathrm{d}R. \tag{42}$$

This is equal to total revenues $I(t) = \tau W(t)N\int_A^\omega S(a)\,(1-F(a))\,\mathrm{d}a$ (see (13)) and thus $E(t) = I(t)$.

D The deduction for general discount rates (section 3.4.1)

I focus on formula pensions that are proportional to $\tau W(t+a)$. Therefore I write $\widehat{P}(R,a,t) = \tau W(t+a)\check{P}(R)$. Furthermore, noting that $W(t+a) = W(t+R)e^{\int_R^a g(s)\,\mathrm{d}s}$ equation (24) can also be written as:

$$\tau W(t+R)\int_R^{R^*}\left(1+\check{P}(R)X\right)e^{\int_R^a g(s)\,\mathrm{d}s}e^{-\delta(a-R)}S(a)\,\mathrm{d}a =$$
$$\tau W(t+R)\int_{R^*}^{\omega}\left(\check{P}(R^*)-\check{P}(R)X\right)e^{\int_R^a g(s)\,\mathrm{d}s}e^{-\delta(a-R)}S(a)\,\mathrm{d}a.$$

For constant wage growth $g(s) = g$ this can be simplified to:[19]

$$\begin{aligned}&\textstyle\int_R^{R^*}\left(1+\check{P}(R)X\right)e^{-(\delta-g)(a-R)}S(a)\,\mathrm{d}a = \\ &\textstyle\int_{R^*}^{\omega}\left(\check{P}(R^*)-\check{P}(R)X\right)e^{-(\delta-g)(a-R)}S(a)\,\mathrm{d}a.\end{aligned} \tag{43}$$

I want to show that for the choice of $\delta = g$ the deductions X coincide with the demographic deduction factor Ψ. I focus first on the NDC system. In this case it holds that $\Psi = 1$ and the conjecture is that $X = \Psi = 1$. Furthermore, $\widehat{P}(R,a,t) = \tau W(t+a)\frac{(R-A)h(R)}{e(R)}$, i.e. $\check{P}(R) = \frac{(R-A)h(R)}{e(R)}$. Noting that $h(R) = \frac{\int_A^R S(a)\,\mathrm{d}a}{(R-A)S(R)}$ and $e(R) = \frac{\int_R^\omega S(a)\,\mathrm{d}a}{S(R)}$ one can thus write: $\check{P}(R) = \frac{\int_A^R S(a)\,\mathrm{d}a}{\int_R^\omega S(a)\,\mathrm{d}a}$. Inserting this into (43) leads to:

$$\left(1+\frac{\int_A^R S(a)\,\mathrm{d}a}{\int_R^\omega S(a)\,\mathrm{d}a}\right)\int_R^{R^*} S(a)\,\mathrm{d}a = \left(\frac{\int_A^{R^*} S(a)\,\mathrm{d}a}{\int_{R^*}^\omega S(a)\,\mathrm{d}a} - \frac{\int_A^R S(a)\,\mathrm{d}a}{\int_R^\omega S(a)\,\mathrm{d}a}\right)\int_{R^*}^{\omega} S(a)\,\mathrm{d}a.$$

This can be simplified to:

$$\textstyle\left(\int_R^\omega S(a)\,\mathrm{d}a + \int_A^R S(a)\,\mathrm{d}a\right)\int_R^{R^*} S(a)\,\mathrm{d}a =$$
$$\textstyle\left(\int_A^{R^*} S(a)\,\mathrm{d}a\int_R^\omega S(a)\,\mathrm{d}a - \int_A^R S(a)\,\mathrm{d}a\int_{R^*}^\omega S(a)\,\mathrm{d}a\right).$$

[19] Similarly, one could also assume a time-varying discount rate $\delta(s)$. Under the assumption that $\delta(s) = g(s)$ one could then derive the same result as in the following.

Collecting terms this leads to:

$$\int_R^\omega S(a)\,\mathrm{d}a \left(\int_R^{R^*} S(a)\,\mathrm{d}a - \int_A^{R^*} S(a)\,\mathrm{d}a \right) = \int_A^R S(a)\,\mathrm{d}a \left(- \int_{R^*}^\omega S(a)\,\mathrm{d}a - \int_R^{R^*} S(a)\,\mathrm{d}a \right).$$

Combining integrals one can conclude:

$$- \left(\int_R^\omega S(a)\,\mathrm{d}a \int_A^R S(a)\,\mathrm{d}a \right) = - \left(\int_R^\omega S(a)\,\mathrm{d}a \int_A^R S(a)\,\mathrm{d}a \right).$$

This proves the conjecture that for $\delta = g$ the implied deduction X equals the demographic deduction Ψ which is just $\Psi = 1$ for the NDC system. Since the demographic deduction factors Ψ_{DB} and Ψ_{AR} are just determined in such a way as to transform the DB and AR systems into a NDC system the same conclusion also holds for these systems.

E Balanced and unbalanced budgets for different discount rates (section 3.4.3)

In this section I study the budgetary implications of the choice of various discount rates. In general, one cannot derive closed form solutions and thus one has to use numerical calculations. For this purpose one also has to make assumptions concerning the density distribution of retirement ages $f(R)$. I work with various simple assumptions in order to highlight the main properties. In section 2.5.3 I have used the elementary case of a two-point distribution with only two possible retirement ages R^L and R^H. Here I use the more general assumption of a triangular distribution that is defined by the minimum and maximum retirement ages R^L and R^H, respectively and also by the mode R^{mod}. The density function is then given by $f(R) = \frac{2(R-R^L)}{(R^H-R^L)(R^{mod}-R^L)}$ for $R^L \leq R \leq R^{mod}$ and $f(R) = \frac{2(R^H-R)}{(R^H-R^L)(R^H-R^{mod})}$ for $R^{mod} \leq R \leq R^H$.[20]

In table A.1 I show three distributions that differ in the shape and the average retirement age $\overline{R} = \frac{R^L+R^{mod}+R^H}{3}$. In all three distributions I assume that the earliest retirement age is given by $R^L = 60$. In the first distribution $R^H = 70$ and the modus, median and mean coincide at $R^{mod} = \overline{R} = 65$. In the second distribution the modus is again $R^{mod} = 65$ while the maximum retirement age is $R^H = 67$ implying a mean of $\overline{R} = 64$. Finally, for the third distribution I assume a non-symmetric case with $\overline{R} = 68$ and $R^{mod} = 67$ which

[20] I have also analyzed different retirement distributions. The results are robust.

Table A.1: Deficit ratios $d = E/I$

	Distribution 1			Distribution 2			Distribution 3		
Type j	$\hat{\delta} = 0$	$\hat{\delta} = 0.02$	$\hat{\delta} = 0.05$	$\hat{\delta} = 0$	$\hat{\delta} = 0.02$	$\hat{\delta} = 0.05$	$\hat{\delta} = 0$	$\hat{\delta} = 0.02$	$\hat{\delta} = 0.05$
DB	1	1	1.004	1	0.982	0.954	1	1.001	1.003
AR	1	1	1.004	1	0.982	0.954	1	1	1.003
NDC	1	1	1.004	1	0.982	0.954	1	1	1.003

Note: The table shows the deficit ratio for three pension schemes, three assumptions of the discount rate $\hat{\delta} \equiv \delta - g$ and three assumed distributions of the retirement age. These are $R^L = 60$, $R^{mod} = 65$, $R^H = 70$ (distribution 1), $R^L = 60$, $R^{mod} = 65$, $R^H = 67$ (distribution 2) and $R^L = 60$, $R^{mod} = 67$, $R^H = 68$ (distribution 3). The mean retirement age is $\overline{R} = 65$ for distributions 1 and 3 and $\overline{R} = 64$ for distribution 2. Mortality follows a Gompertz distribution with $\alpha = 0.000025$ and $\beta = 0.096$ and the target retirement age is always $R^* = 65$.

implies an average retirement age of $\overline{R} = 65$.

The first result is that the budget is in balance for all three distributions of retirement ages as long as the discount rate is equal to the growth rate of wages (i.e. $\hat{\delta} = 0$). This is of course an expected result that follows from the analytical findings of section 3.4.1. As a second result one can see that the budget is also (approximately) balanced for situations where $\hat{\delta} > 0$ as long as $R^* = \overline{R}$ (which is the case for distributions 1 and 3). For the second distribution, however, for which $R^* = 65 > 64 = \overline{R}$ this is different. In this case the pension system runs a permanent surplus if the discount rate is above the growth rate of wages ($d = 0.98$ for $\hat{\delta} = 0.02$ and $d = 0.95$ for $\hat{\delta} = 0.05$).