A rank-dependent utility model of uncertain lifetime*

Nicolas Drouhin†

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Abstract

In a continuous time life cycle model of consumption with an uncertain lifetime, we use a non-parametric specification of rank-dependent utility theory to characterize the preferences of the agent. We prove that time consistency holds for a subclass of probability-weighting function, providing the foundation for a constant rate of time preference that interacts multiplicatively with the hazard rate instead of additively as in Yaari (1965) seminal model. We calibrate both models to explain the hump in the life-cycle consumption, and show that the multiplicative model is more robust.

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A long tradition exists in economics that distinguishes two types of primitive to explain discounting. First, discounting is explained using purely psychological factors, such as impatience, captured by the discount function. If the discount function is exponential, as in the seminal model proposed by Samuelson (1937), then the time preference is characterized by a "pure rate of time preference" (*i.e.* the log derivative of the discount function) that is invariant with time and the level of consumption. Even if some authors considered early on the possibility for the discount factor to be "non-exponential" (for example Yaari, 1964; Harvey, 1986, 1995), only with the behavioral revolution were alternative *ad hoc* parametrical discount functions proposed, and used systematically in the applied economics literature. Among them, the "quasi hyperbolic" discount function (Phelps and Pollak, 1968; Laibson, 1997) is probably the most popular.

The second explanation for discounting is simply to consider that future prospects are uncertain. In this case, considering that the utility of future prospects are weighted according to their probability of being effectively consumed at the given date is reasonable (refer to Sozou, 1998; Dasgupta and Maskin, 2005, for a general discussion of that topic). Among this literature, models of intertemporal choice with uncertain

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†Corresponding author at Ecole Normale Supérieure de Cachan and Centre d'économie de la Sorbonne UMR CNRS 8174 e-mail: nicolas.drouhin@ens-cachan.fr

lifetimes are good tools to investigate the theory of discounting. The seminal paper by Yaari (1965) considered expected utility maximizers with known probability distributions of the"age of death", and a standard exponential discounting life-cycle utility. More recent models considered various types of sophisticated utility frameworks to address lifetime uncertainty. For example, Moresi (1999) considered an application of the "ordinal certainty equivalent hypothesis" (Selden, 1978). Bommier (2006, 2013) considered a concave transformation of the life-cycle utility to explain "risk aversion with respect to length of life". Halevy (2008) used the"dual theory of choice" (Yaari, 1987). Ludwig and Zimper (2013), Groneck et al. (2012) and d'Albis and Thibault (2012) considered an ambiguous survival probability.

In this study, we build a model of the intertemporal choice of consumption and saving with an uncertain lifetime in which the agent psychologically transforms her survival probability distribution, such as in the *rank-dependent utility model* (Quiggin, 1982), or in the *cumulative prospect theory* (Tversky and Kahneman, 1992). The idea of introducing a rank-dependent utility in this setting was explored by Drouhin (2001) and Bleichrodt and Eeckhoudt (2006). The originality of this study is that we use continuous time modeling and optimal control to solve the model. With this methodology, we are able to discuss the important topic of time consistency, the main criteria of rationality over time. Following Strotz (1956), there exists a conventional wisdom in economics that considers that any departure from exponential discounting implies time inconsistency. When considering uncertain prospects, part of the literature focuses on the related notion of dynamic consistency (refer to Machina, 1989; Etchart, 2002; Halevy, 2004a,b; Nebout, 2014, for a discussion).

The main result of the study is that any agent who transforms the probability distribution of the age of death with a power function is time consistent. This result provides a foundation for a rate of time preference that interacts multiplicatively with the probability of dying instead of additively in the standard expected utility approach. On empirical grounds, a calibrated version of the model shows that the multiplicative rate of time preference has better properties than the traditional additive rate of the exponential discounting model. This multiplicative rate allows for the solution to certain paradoxes of the literature (the hump of life-cycle consumption, excessive sensitivity to variations in the interest rate).

The remainder of this study is as follows. Section 1 presents the intertemporal utility functional used. Section 2 solves the model in the absence of life annuities. Section 3 discusses time consistency and provides foundations for the multiplicative model of the rate of time preference. Section 4 discusses a calibrated parametrical version of the model on empirical grounds. Section 5 discusses the model when the agent has access to life annuities. Finally section 6 concludes.

1 A Rank-Dependent Utility model of consumption and savings with an uncertain lifetime

We consider an agent's choice of her consumption profile. A consumption profile is a function of time defined on the interval $[0, T]$, with 0 representing the age of birth and

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T being an arbitrary constant, interpreted as the maximum possible life duration for the agent. Because we are interested in understanding the manner in which the timing of the decision influences the choice of the consumption profile, we denote the date of the decision by $t \in [0, T)$.
In a first step, we consider the case in which the agent, alive on age t, knows with certainty her age at death, s. We make the following assumptions:

A1 If an agent knows with certainty her age of death s, her intertemporal preferences are represented by an intertemporal utility functional assumed to be additive:

$$V_t(c, s) = \int_t^s F(\tau - t) u(c(\tau)) \, d\tau \tag{1}$$

with $\lim_{c \to 0} u(c) = +\infty$, $u'(c) > 0$ and $u''(c(\tau)) < 0$.

A2 $\forall \tau \in [t, T], F(\tau - t) > 0$, $F'(\tau - t) \leq 0$

A3 (monotonicity according to lifespan). $\forall c : \quad s' > s \Rightarrow V_t(c, s') > V_t(c, s)$

A1 is the standard assumption for a life cycle model with a certain lifetime. This assumption guarantees the existence of a strictly positive consumption profile throughout the life cycle.

$F(\tau - t)$ is the *riskless discount factor*. When it is strictly decreasing the agent exhibits preference for present consumption.

A1, A2 and A3 together imply that the *per period felicity*, u, and the *intertemporal utility functional*, V, are both necessarily positive in their domain.

A3 indicates that, for a given consumption profile, outcomes are always ranked according to lifespan. When introducing uncertainty, our model is a natural candidate for using a rank-dependent utility.

The agent actually does not know with certainty her age of death.
In the remaining, we will use the letters s or τ for the age of the agent, when it considered as a variable. For some critical ages of the life-cycle, we will use a variation of the letter t (t, is "the age at which the agent plan her future consumption", T is the "maximum possible age", t_R will be "the age of retirement", etc.).
We assume that for a living agent at age t, the age of death $s \geqslant t$ is an absolutely continuous random variable defined on the interval $[t, T]$. We denote by $\pi_t(s) > 0$ the probability density function of this random variable and $\Pi_t(s)$, the cumulative distribution function. Thus, we have:

$$\Pi_t(s) = \int_t^s \pi_t(\tau) \, d\tau$$

with $\Pi_t(T) = 1$.
$\Pi_t(s)$ is interpreted as "the probability of being dead at age s, knowing you are alive at age t", and, $(1 - \Pi_t(s))$ as "the probability of being alive at age s, knowing you are alive at age t". The usual rules concerning conditional probability apply:

$\forall s \geqslant t'$

and

$$\pi_{t'}(s) = \frac{\pi_t(s)}{1 - \Pi_t(t')} \tag{3}$$

In the special case in which $s = t'$, we obtain the *hazard rate* at age s:

$$\pi_s(s) = \frac{\pi_t(s)}{1 - \Pi_t(s)} \tag{4}$$

Thus, we face a special problem of choice under uncertainty. If we assume that the agent is an expected utility maximizer, with an additively separable utility functional as in Yaari (1965), we have the following:

$$\begin{aligned} EV_t(c) &= \int_t^T \int_t^s F(\tau - t)u(c(\tau)d\tau \ d\Pi_t(s) \\ &= \int_t^T \int_t^s F(\tau - t)u(c(\tau)d\tau \ \pi_t(s)\, ds \end{aligned} \tag{5}$$

We depart from this model by using a more general model of choice under uncertainty, the *rank-dependent utility* model, introduced by Quiggin (1982, 1993), and popularized by Tversky and Kahneman (1992) as an important part of their *cumulative prospect theory*. This model has many very interesting properties: it introduces probability transformation but in a manner that preserves first-order stochastic dominance; it provides a solution to the so-called Allais paradox (Allais, 1953)[1]; and when time is involved, it disentangles risk aversion from resistance to intertemporal substitution (the log-derivative of the *per period felicity function*).

The idea is very simple. We simply replace the cumulative distribution function, Π_t, by an increasing transformation of it, $h(\Pi_t)$, in the expected utility model given by equation (5).

$$\begin{aligned} RDU_t(c) &= \int_t^T \int_t^s F(\tau - t)u(c(\tau)d\tau \ dh(\Pi_t(s)) \\ &= \int_t^T \int_t^s F(\tau - t)u(c(\tau)d\tau \ h'(\Pi_t(s))\pi_t(s))ds \end{aligned} \tag{6}$$

with h being a probability weighting function assumed to be continuous and twice differentiable and such that: $h(0) = 0$, $h(1) = 1$ and $h' > 0$.
Notice that (5) is a special case of (6) when $h(\Pi_t(s)) = \Pi_t(s)$. Integrating (6) by parts, we obtain:

$$RDU_t(c) = \int_t^T (1 - h(\Pi_t(s))F(\tau - t))\, u(c(s))\, ds \tag{7}$$

For the agent, the expected present value at age t of the utility stream between t and T is the integral over this interval of the product of the utility of consumption at each age s of the interval with the subjective weight given by the agent to the event

[1]For readers unaware of this literature Appendix A provides a quick introduction to the paradox and its solution within RDU.

"being alive at age s". Equation (7) makes explicit our initial intuition within the continuous time framework. The factor $f_t(s) = (1 - h(\Pi_t(s)))\, F(\tau - t)$ is the *effective discount factor* applied to utility of the consumption at age s viewed from age t. This factor depends on the probability distribution of the ages of death and the subjective transformation of this probability distribution, and on the *riskless discount factor*. The *effective discount factor* is continuous, differentiable and strictly decreasing from one to zero on the interval $[t, T]$.

Taking the log-derivative of the effective discount factor, we also define the *effective rate of discount* of utility at age s viewed from age t:

$$\theta_t(s) \stackrel{\text{def}}{=} \pi_t(s) \frac{h'(\Pi_t(s)))}{1 - h(\Pi_t(s))} + \frac{F'(s-t)}{F(s-t)} \tag{8}$$

The *effective rate of discount* is the sum of two terms. The first term stems from lifetime uncertainty, and, following Drouhin (2001), it can be decomposed in two factors. The first factor, $\pi_t(s)$, the probability density associated with the event "dying at age s, knowing that your alive at age t", is interpreted as the *objective part of time preference*. The second factor, $h'(\Pi_t(s))/(1 - h(\Pi_t(s)))$, depends on the manner in which the agent transforms probability distributions and its value depends on her preferences. For that reason, this second factor is qualified as *subjective*[2]. This factor stems from the use of a rank-dependent formulation and is the main innovation of this article. Thus, it is of first importance when discussing the consequences of this formulation for standard results on time consistency and life insurance in the following section. Finally, and more classically, the second term of the *effective rate of discount* is the *riskless rate of time preference*, which is also subjective and which intensity does not depend on lifetime uncertainty. In the literature, this second term is frequently called *pure rate of time preference.*

Now that the *effective rate of time preference* is defined, rewriting the rank-dependent intertemporal utility functional (7) as follows is convenient:

$$RDU_t(c) = \int_t^T \exp\left(\int_s^t \theta_t(\tau)\, d\tau\right) u(c(s))\, ds \tag{9}$$

We now investigate the properties of the choice of the optimal consumption path made by an agent on age t.

2 Optimal consumption path with no life annuities

To express the optimal consumption path, we first define the feasible set of consumption profiles. We assume that, at each age s, the living agent receives a flow of non-financial income $w(s)$, assumed to be strictly positive and differentiable, and a flow of financial income proportional to her assets $a(s)$. These incomes are either used for current consumption or saved for future consumption. At this stage, we assume that no life annuities or insurance exists. The only assets available for savings are standard bonds,

[2]Hurd et al. (1998) empirically investigate the idea that an agent can subjectively transform her survival probability distribution.

that earn a constant rate of interest, r. Thus, at each time $s \in [t, T]$, the standard intertemporal budgetary constraint holds:

$$\forall s \in [0, T], \qquad \dot{a}(s) = w(s) + ra(s) - c(s) \tag{10}$$

We also assume that no "bequest motive" exists, implying that an agent living her maximum possible life-duration chooses to leave no bequest ($a(T) = 0$). Thus, if we sum on the interval $[t, T]$ the differential constraints (10) at each age weighted by the economical discount factor $\exp(-r(s-t))$, after some simple manipulations, we get:

$$a(t) + \int_t^T w(\tau) e^{-r(\tau - t)} d\tau = \int_t^T c(\tau) e^{-r(\tau - t)} d\tau \tag{11}$$

This is the very standard *life cycle budgetary constraint*, the present value of all incomes over the life cycle is equal to the present value of the consumption stream.

Using the same method, but integrating over the sub-interval $[t, s] \subset [t, T]$, we can express the level of asset at any age :

$$a(s) = a(t)\, e^{r(s-t)} + \int_t^s (w(\tau) - c(\tau))\, e^{r(s-\tau)} d\tau \tag{12}$$

In this section we do not consider any *borrowing constraint*. Section 4 introduce this constraint and discusses it extensively.

We denote $c_t(s)$ as the optimal consumption path decided at age t for the time interval $[t, T]$. Thus $c_t(s)$ is the solution to the following program:

$$\mathcal{P}_t \quad \begin{cases} & \max_c RDU_t(c) = \int_t^T (1 - h(\Pi_t(s)))\, F(s-t) u(c(s))\, ds \\ \text{s.t.} & \forall s \in [t, T), \dot{a}(s) = w(s) + r\, a(s) - c(s)) \\ & a(t) = cst \\ & a(T) = 0 \end{cases}$$

Because of the continuity of h, w, Π_t and the continuity and strict concavity of u this program is shown to admit a unique solution that is continuous and differentiable. By applying *Pontryagin's maximum principle*, the resolution of such a program implies to solve a system of differential equations. If, for not losing generality of the results, we refuse to specify a special "easy to use" functional form for the *per period felicity function*, earnings function and probability distribution of the age of death, the only thing we can do is to derive the rate of growth of the optimal consumption path planned at age t.

Denoting $\gamma_t(s)$ as the *coefficient of relative resistance toward intertemporal substitution*,

$$\text{with} \quad \gamma_t(s) \stackrel{\text{def}}{=} -\frac{u''(c_t(s))}{u'(c_t(s))} c_t(s) \tag{13}$$

we have :

Proposition 1. *Without life annuities, at each age s, the rate of growth of the optimal consumption path planned at age t is:*

$$G(t,s) = \frac{\dot{c}_t(s)}{c_t(s)} = \frac{r - \theta_t(s)}{\gamma_t(s)} \tag{14}$$

and the consumption function is :

$$c_t(s) = c_t(t) \exp\left(\int_t^s G(t,\tau)d\tau\right) \tag{15}$$

Proof: The Hamiltonian of agent's program is:

$$H(c(s), a(s), \lambda(s), s) = (1 - h(\Pi_t(s))) F(s-t) u(c(s)) + \lambda(s)(w(s) + r\, a(s) - c(s))$$

First order conditions give:

$$\frac{\partial H}{\partial c} = 0 \Rightarrow \lambda(s) = (1 - h(\Pi_t(s))) F(s-t) u'(c_t(s)) \tag{16}$$

$$\frac{\partial H}{\partial a} = -\dot{\lambda}(s) \Rightarrow \dot{\lambda}(s) = -r\lambda(s) \tag{17}$$

Taking the logarithm of (16) and differentiating according to s we get:

$$\frac{\dot{\lambda}(s)}{\lambda(s)} = -\frac{h'(\Pi_t(s))\,\pi_t(s)}{1 - h(\Pi_t(s))} + \frac{F'(s-t)}{F(s-t)} + \frac{u''(c_t(s))}{u'(c_t(s))}\frac{dc}{ds}(s) \tag{18}$$

Collapsing the equations, and using definition (8), we deduce equation (14). Differentiating equation (15), we easily retrieve equation (14). □

The rate of growth of the consumption path is the difference between the rate of interest (economic discount rate) and the effective rate of time preference, both divided by an index of the curvature of the utility function usually referred as the coefficient of relative risk aversion or, more properly in this context and according to Gollier (2001), as the resistance to intertemporal substitution. The important point is that, as in Yaari (1965) the effective rate of discount is no longer constant and provides a wide variety of possible dynamics for consumption. However, in contrast to Yaari (1965), not only the properties of the probability distribution of the ages of death matter. The manner in which agents subjectively transform this probability distribution also matters. If we want to go further, we must specify additional restrictions for the model, which is done from a normative point of view in the next section, and from a descriptive point of view in the subsequent section.

3 Normative point of view: Time consistency

The intertemporal choice model with uncertain lifetime combines both risk and time. We can specify the model for being consistent with some criteria of rationality. Because the model uses rank-dependent utility, it fulfills necessarily and by construction the main axiom of rationality toward risk, *first-order stochastic dominance* (Quiggin, 1993). According to decisions related to time, the main criterion of rationality is *time consistency* as proposed by Strotz (1956). What restriction do we have to impose on the probability transformation function to fulfill time consistency? To answer this question, we must properly define the notion of *time consistency.* As in Strotz (1956), Caputo (2005), and Drouhin (2009, 2012), we use a "choice-based" methodology that enable us to compare agents' effective decisions on different dates, meaning that we consider the solution to maximization programs under constraints.[3] In the absence of new information, an agent is said to be time consistent if she behaves in the future as she planned to in the past.

Definition 1 (Time consistency)**.** *We denote $c_t$and a_t the solution to program $\mathcal{P}_t$. If we denote $c_{t'}$and $a_{t'}$, the optimal solution to program $\mathcal{P}_{t'}$, with $t' \in [t, T]$ and:*

$$\mathcal{P}_{t'} \quad \begin{cases} \max\limits_{c} RDU(c) = \displaystyle\int_{t'}^{T} \left(1 - h\left(\Pi_t(s)\right)\right) F(s-t') u\left(c(s)\right) ds \\ u.c. \qquad \dot{a}(s) = w(s) + r\ a(s) - c(s)) \\ \qquad\qquad a(t') = a_t(t') \\ \qquad\qquad a(T) = 0 \end{cases}$$

then an agent is time consistent *if and only if:*

$$\forall t \in [t_0, T]\,,\ \forall t' \in [t, T]\,, \forall s \in [t', T]:\ c_t\left(s\right) = c_{t'}\left(s\right) \tag{19}$$

A corollary to this definition is that, to be time consistent, the rate of discount at each age s must be independent from the decision age t.[4] In the special case of expected utility with no "pure time preference", $\theta_t\left(s\right) = \pi_s(s)$. Whatever the form of the probability distribution, the rate does not depend on the planning decision age, it is "time distance independent", and time consistency holds. However, for other cases, the distribution probability of the age of death and the rank-dependent utility provide a special mathematical structure for the effective discount rate and factor. We notice that, in the most general case $\theta_t\left(s\right)$ is time-distance dependent because it depends on $\Pi_t\left(s\right)$, a strong presumption for time inconsistency. Nevertheless are there some other cases for which time consistency holds?

Proposition 2. *An agent is time consistent if and only if her probability distribution transformation function is of the form $h\left(x\right) = 1 - \left(1 - x\right)^{\alpha}$ with $(\alpha > 0)$ <u>and</u>, the discount factor is such that $\forall t' \in [t, T], \forall s \in [t', T]$, $F(s - t') = \delta(t') \exp[-\beta(s - t)]$ with β as a constant and $\delta(t')$ as an arbitrary function. In this case:*

$$\forall t' \in [t, T], \forall s \in [t', T], \theta_{t'}\left(s\right) = \theta\left(s\right) = \alpha\, \pi_s\left(s\right) + \beta \tag{20}$$

Proof:

(sufficiency)

Considering (20) and proposition 1, we have:

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[3] Some articles use an "axiomatic" or "preferences-based" definition of time consistency (Ahlbrecht and Weber, 1995; Moresi, 1999, for example).

[4] Refer to Drouhin (2012) for a proof.

$\forall t' \in [t, T]$, $\forall s \in [t', T]$, $\frac{\dot{c}_t(s)}{c_t(s)} = \frac{\dot{c}_{t'}(s)}{c_{t'}(s)}$.
Considering the life cycle budgetary constraint of programs $\mathcal{P}_t$ and $\mathcal{P}_{t'}$ we deduce that $a_t(t') + \int_{t'}^{T} w(s)e^{-r(s-t')}ds = \int_{t'}^{T} c_t(s)e^{-r(s-t')}ds$ and $a_{t'}(t') + \int_{t'}^{T} w(s)e^{-r(s-t')}ds = \int_{t'}^{T} c_{t'}(s)e^{-r(s-t')}ds$. By definition of program $\mathcal{P}_{t'}$ we have $a_{t'}(t') = a_t(t')$. Thus, we have: $\int_{t'}^{T} c_t(s)e^{-r(s-t')}ds = \int_{t'}^{T} c_{t'}(s)e^{-r(s-t')}ds$.
The two functions have the same slope on the interval $[t', T]$ and the same discounted integral. That implies that they are equal all over the interval.

(necessity)
If the agent is time consistent, she fulfills equation (19). A strictly positive and differentiable c_t implies that:

$$\forall t \in [0, T]\,,\ \forall t' \in [t, T]\,, \forall s \in [t', T] :\ \frac{\dot{c}_t(s)}{c_t(s)} = \frac{\dot{c}_{t'}(s)}{c_{t'}(s)}$$

Taking into account(14), (8), and (3) implies that:
$\forall t \in [0, T]\,,\ \forall t' \in [t, T]\,, \forall s \in [t', T] :$

$$\frac{h'\left(\Pi_t\left(s\right)\right)}{1 - h\left(\Pi_t\left(s\right)\right)} = \frac{h'\left(\Pi_{t'}\left(s\right)\right)}{\left(1 - h\left(\Pi_{t'}\left(s\right)\right)\right)\left(1 - \Pi_t\left(t'\right)\right)} \tag{21}$$

and

$$\frac{F'(s-t')}{F(s-t')} = \frac{F'(s-t)}{F(s-t)} \tag{22}$$

Equation (21) should hold in the particular case in which $s = t'$. Considering this case and noting that $\Pi_t\left(t\right) = 0$, we obtain:

$$\forall t \in [0, T]\,,\ \forall t' \in [t, T] : \frac{h'\left(\Pi_t\left(t'\right)\right)}{1 - h\left(\Pi_t\left(t'\right)\right)} = \frac{h'\left(0\right)}{\left(1 - \Pi_t\left(t'\right)\right)} \tag{23}$$

This first-order differential equation has a set of solutions fully described by $h\left(x\right) = 1 - \left(1 - x\right)^{\alpha}$, with $\alpha = h'\left(0\right)$.
For Equation (22) to hold F should be multiplicatively separable in t and s. The only way to achieve this result is to have an exponential function of the form $F(s - t') = \delta(t')\exp[-\beta(s - t)]$. □

Corollary 2.

a. When $\alpha > 1$, h is strictly concave.

b. For time consistent agents with the same per period felicity function*, risk aversion in the sense of Bommier et al. (2012) increases with parameter α.*

Proof:

2-a is obvious: $h''(x) = -(\alpha - 1)\alpha(1 - x)^{\alpha}$

2-b Bommier et al. (2012) worked with a formulation of the rank-dependent utility theory using a probability weighting function based on the decumulative distribution function ϕ. We have: $1 - \phi(1 - x) = h(x) \Rightarrow \phi(1 - x) = (1 - x)^{\alpha}$. The convexity of ϕ increases with α. Then, we simply apply Bommier et al. (2012), p. 1628. □

Corollary 2 is important for choosing a "realistic" value for α. As was already stated, a concave probability transformation offers a solution to the Allais paradox.

Thus, a value of $\alpha > 1$ seems realistic and implies that agents show more risk aversion than in the standard expected utility model.

Some points should be emphasized.

1. The expected utility model is not the only model compatible with time consistency. The rank-dependent utility model with a power function for transforming the probability distribution[5] also implies time consistency, which provides behavioral foundations for a model of intertemporal choice that is different from the original discounted expected utility model.
   Considering the effective rate of discount of a time-consistent agent given by Equation (20), the intertemporal utility functional can be rewritten as:
$$RDU_t(c) = V_t(c) = \int_t^T e^{\int_s^t (\alpha\pi_\tau(\tau)+\beta)d\tau} u(c(s))\, ds \tag{24}$$

2. In our model, as in Yaari (1965), lifetime uncertainty is a primitive of time preference. The hazard rate $\pi_\tau(\tau)$ is the main component of the rate of discount. We name this rate the "objective" component of time preference, which indicates that the main reason for an existing bias in favor of present consumption is that we know that we are mortal, but we do not know when we will die; we simply have an idea of the distribution probability of the age of death. However, the fact that time preference has an *objective component* does not mean that it does not have a *subjective component*. In our model, this component is captured by two constant parameters.
   β is the traditional *pure rate of time preference*. It is named "pure" because this type of time preference is independent of the probability of dying, indicating that with $\beta > 0$, even an "immortal agent" shows preference for present consumption. In the remainder of this study, we will call β the *additive rate of time preference* to distinguish it from α, which is called the *multiplicative rate of time preference*. This rate is the main innovation of this article. If $\alpha = 1$ our model is the standard Yaari (1965) model. When $\alpha > 1$, this means that the agent gives a psychological weight to present consumption more important than the instantaneous probability of dying. In this case the agent demonstrates *relative preference for present consumption*. In RDU/ cumulative prospect theory, the behavior is interpreted as "pessimistic" in the sense that the agent tends to overweight her probability of dying.[6] In the opposite situation, when $\alpha < 1$, the agent demonstrates a *preference for future consumption* and underweights her probability of dying ("optimistic" behaviour). It is important to precise that the use of the terms "optimism" and "pessimism" should not be interpreted as irrationality. As previously stated, agents with preferences represented by the utility functional (24) fulfills transitivity, first-order stochastic dominance, and time consistency.

3. Thus, the generalisation of Yaari (1965) is characterised by the existence of two parameters, instead of one, that capture time preference. According to Occam's

[5]Diecidue et al. (2009) provided axiomatic foundations for such probability transformation function.

[6]Refer to Wakker (2010), 172-176, for an extensive discussion of probability transformation as pessimism/optimism.

razor principle, a scientist should not multiply parameters without necessity. Therefore, for a better understanding of the innovation of this study, comparing two polar cases of the generalized model with each case characterized by only one parameter is useful. The first case is characterized by $\alpha = 1$ and $\beta \neq 0$. This model is identical with the one by Yaari (1965). We call it the *model of additive rate of time preference*. In this case the intertemporal preferences of the agents are represented by:

$$V_t^+(c) = \int_t^T e^{\int_s^t (\pi_\tau(\tau)+\beta)d\tau} u(c(s))\, ds \tag{25}$$

The second case is characterized by $\alpha \neq 1$ and $\beta = 0$, which we call the *model of multiplicative rate of time preference*. In this case, the intertemporal preferences of the agents are represented by:

$$V_t^\times(c) = \int_t^T e^{\int_s^t \alpha\pi_\tau(\tau)d\tau} u(c(s))\, ds \tag{26}$$

4. To understand the potential of the *multiplicative rate of time preference*, considering the special case of an infinite horizon *i. e.* T being infinite, can be interesting. In a first step, we assume that for all $s \in \mathbb{R}^+$, $\pi_s(s) = \pi = cst.$ In this case the intertemporal utility functional is equivalent to the exponential discounting model (with $\alpha\,\pi + \beta = \theta$):

$$RDU_t(c) = \int_t^{+\infty} e^{-\theta(s-t)} u(c(s))\, ds \tag{27}$$

If we adopt a more realistic model of the uncertain lifetime, such as for example the Gompertz or the Gompertz-Makeham law of mortality, then the hazard rate increases with age, which has an important consequence as noted in the following proposition.

Proposition 3. *If the consumption stream is bounded and age $\hat{t}$ exists such that for all $t > \hat{t}$, $\frac{\partial \pi_t(t)}{\partial t} > 0$, then, for all $T \in \mathbb{R}^+ + \{+\infty\}$, the intertemporal rank-dependent utility functional* (24) *is always definite.*

Proof: Obvious. □

This proposition indicates that our concept of a *multiplicative rate of time preference* is more robust than the usual notion of pure time preference from the exponential discounting model. In particular a preference for present consumption in our model ($\alpha > 1$) is not a prerequisite for the intertemporal utility functional to be definite, even when the horizon is infinite. From a behavioral perspective, this statement indicates that our model is a tool to explore possibilities than cannot be addressed using the standard discounted expected utility model.

5. As we have already noted, uncertain lifetime is an fundamental characteristic of human condition that is sufficient to explain "preference for present consumption". However, when economists started to model intertemporal choice they

lacked the mathematical skills to build a model that accounts for this uncertainty. Therefore, they invented the *ad hoc* notion of *pure time preference* to explain the observed behavioral bias in favor of present consumption. As was just shown, the exponential discounting model is an approximation of the model proposed in this study. However, as a thought experiment, imagine that with all the mathematical skills we have today, we can go back in the past to be the first to model intertemporal choice of consumption and savings. As a primitive for time preference, we can choose to incorporate in our model an assumption of uncertain lifetime or an assumption of "pure time preference" or both assumptions as we did in the beginning of this paper. If we seriously consider Occam's razor principle, which states that *pluralitas non est ponenda sine necesitate*, then we have, in a first step, to choose only one primitive and the more obvious of the two. In this case, we seem to need to eliminate the assumption of a "pure time preference". The consequence is clear, the *multiplicative model of time preference* should be considered as the basic model of intertemporal choice and, the introduction of the notion of "pure time preference" should only be considered, in a second step, if the basic model fails to fit the data. Unfortunately, the path followed by the economic theory of intertemporal choice has been very different from the one we just imagine. Education and training have so deeply imbedded the notion of "pure time preference" in the mind of the economist that accepting the possibility of discarding it requires significant effort.

In the next section, we compare the multiplicative and the additive models of the rate of time preference in the context of a realistic simulation of agent choice.

4 Descriptive point of view: Explaining the life-cycle hump in consumption

To better understand the differences between the additive and the multiplicative models of time preference, running numerical simulations with a parametrical version of the model is useful. In particular observing how two versions of the model reproduce certain stylized facts and paradoxes of the empirical literature on consumption and savings may be interesting.

At least since the work of Thurow (1969), that *the intertemporal consumption profile is hump shaped* with a maximum consumption between the ages of 45 and 50 is well known (see Attanasio and Weber, 2010, for a recent and complete survey). This result is paradoxical in regard to the basic model of the intertemporal choice of consumption and savings, because of the independency between the slope of the consumption profile and age in this model.[7]

Following, the intuition in Thurow (1969), the main line of the literature attempts to solve the paradox relying on the co-existence of a borrowing constraint and a negative slope for the optimal unconstrained consumption profile. Because the slope of the income profile is positive, an unconstrained agent wants to borrow in the beginning of

[7]Another related paradoxical observation is the rapid decline of consumption in the vicinity of the age of retirement (see Battistin et al., 2009, for a recent introduction to this topic).

her life. Because she is not allowed to do so, her consumption matches her income at least during the first part of her working life. This effect is amplified if labor income is uncertain, adding to the model a precautionary motive for savings that is stronger than impatience for young agents (Carroll, 1997). Among others, Gourinchas and Parker (2002) empirically showed that those *buffer-stock savings* models can explain the hump in the consumption profile. Other explanations also exist. For example, following Heckman (1974), if consumption and leisure are substitute, agents work more during middle age when productivity and wages are higher, and they are compensated by higher consumption during these ages. Thus, without market imperfections, one can obtain a life-cycle consumption profile that follows the productivity and earnings profiles, and thus is hump shaped.

Of course, because the life-cycle model of intertemporal choice with uncertain lifetime implies an age-increasing *effective rate of discount*, it is a good candidate for solving the paradox. Strangely, only a few articles explore this relation ((Bütler, 2001; Hansen and Imrohoroglu, 2008; Feigenbaum, 2008). The latter remains a bit skeptical on the possibility for uncertain lifetime to explains the hump on its own, because of the necessity for a very specific set of parameters. In particular Feigenbaum (2008), p. 863, argued that, given the necessity of a high intertemporal elasticity of substitution[8] to fit the data, in general equilibrium a small change in the rate of interest has a dramatic effect on the age and size of the hump.

However, from a theoretical point a view, the preceding reasoning, is conditioned by the special form for the intertemporal utility functional chosen by the author. That is precisely the point of Bommier (2013), that accounts for the hump in a non standard model of intertemporal choice with an uncertain lifetime. The model remains in the expected utility framework but departs from Yaari (1965) seminal model by using a life-cycle utility functional that is stationary, but no longer additively separable. Bommier (2013) built on previous work (Bommier, 2006, 2007) and insist on the role of *intertemporal correlation aversion* (ICA) to model intertemporal choice. Many authors insisted on the fact that the hazard rate is too low at the age of the hump to explain it. Bommier (2013) showed that, because ICA implies risk aversion with respect to life duration, it magnifies the impact of the probability of dying. Because Bommier (2013) calibrated his model in the case in which the agent can purchase perfect annuities, we will retrun to his model in the next section.

Our theoretical approach is a generalization of the standard model, that is symmetric from the one of Bommier[9]. As was shown, we keep additive separability of the life-cycle utility functional when life duration is certain, but we relax the assumption of expected utility to the more general rank-dependent utility approach. Our model of the multiplicative rate of time preference, when calibrated to fit the empirical characteristics of the hump in life-cycle consumption solves the problem of robustness to change in the parameters raised by Feigenbaum (2008). However, before proving that, we have to make some complementary assumptions to render the model eligible such an empirical calibration.

[8] $1/\gamma$ in our model.

[9] Of course the approach of Bommier (2013) and the one of this paper could be merged in an even more general model

The first modification that we make is to choose a more specific *per period felicity function.* Because we were interested in obtaining the most general theoretical results, until now we have worked with a *per period felicity function* that was only assumed to be positive, increasing and concave. When coming to numerical simulations, the cost of this general formulation is that the consumption function can only be defined implicitly, as shown in Proposition 1, equation (15). For an explicit formulation of the consumption function, then we must assume that the *per period felicity function* is a CES function ($u(c) = c^{(1-\gamma)}/(1-\gamma)$, with $\gamma \in (0,1) \cup (1,+\infty)$. The case $\gamma = 1$, which corresponds to a *logarithmic per period felicity function* is excluded from the analysis by assumption A3, which imposes positivity over the entire domain of the *per period felicity function.*

Thus the two polar cases of additive and multiplicative time preference are rewritten as follows:

$$V_t^{+}(c) = \int_t^T \exp\left(\int_s^t (\pi_\tau(\tau) + \beta)\, d\tau\right) \frac{c(s)^{1-\gamma}}{1-\gamma}\, ds \tag{28}$$

$$V_t^{\times}(c) = \int_t^T \exp\left(\int_s^t \alpha\, \pi_\tau(\tau) d\tau\right) \frac{c(s)^{1-\gamma}}{1-\gamma}\, ds \tag{29}$$

The second modification that must be made is to explicitly account for the *borrowing constraint.* As suggested by Yaari (1965), in a world with no life insurance, an agent with uncertain lifetime finds no one who will lend her some money. Therefore, for all $s \in [t,T), a(s) \geq 0$. Leung (1994, 2001, 2007) emphasized the impossibility of having an interior solution over the entire domain of the consumption function. Under very general conditions, a non empty final interval of ages $[t^*, T]$ always exists in which the constraint binds.

The last required modification is to consider the possibility of the income profile as discontinuous at the age of retirement.[10]

The agent's program is rewritten as:

$$\mathcal{P}'_t \quad \begin{cases} & \max_c V_t(c) \\ \text{s.t.} & \forall s \in [t,T), \dot{a}(s) = w(s) + r\, a(s) - c(s)) \\ & \forall s \in [t,T), a(s) \geq 0 \\ & a(t) = cst \\ & a(T) = 0 \end{cases}$$

The Hamiltonian of the agent's program is:

$$H(c(s), a(s), \lambda(s), s) = \exp\left(\int_t^s -\theta(\tau) d\tau\right) u\left(c\left(s\right)\right) + \lambda\left(s\right)\left(w\left(s\right) + r\, a\left(s\right) - c_t\left(s\right)\right)$$

To address the borrowing constraint, we formulate the generalized Lagrangian of the problem:

$$\mathcal{L}(c(s), a(s), \lambda(s), \mu(s), s) = H(c(s), a(s), \lambda(s), s) + \mu(s) a(s)$$

[10]Because of piecewise continuity and differentiability, the "dot" notation refers now to the right derivative of the function of time used, *i.e.* $\dot{a}(s) = \lim_{\tau \to s^+} \frac{da}{d\tau}(\tau)$

First-order conditions give:

$$\frac{\partial \mathcal{L}}{\partial c} = 0 \Rightarrow \lambda\left(s\right) = \exp\left(\int_{t}^{s} -\theta(\tau)d\tau\right) u'\left(c\left(s\right)\right) \tag{30}$$

$$\frac{\partial \mathcal{L}}{\partial a} = -\frac{d\lambda}{ds}\left(s\right) = -\dot{\lambda}(s) \Rightarrow \dot{\lambda}(s) = -r\lambda(s) - \mu(s) \tag{31}$$

$$a(s) \geq 0 \tag{32}$$

$$\mu(s) \geq 0 \tag{33}$$

$$\mu(s)a(s) = 0 \tag{34}$$

For a given intertemporal utility functional V, we denote $I_i^V \subset [t, T]$, all the time intervals over which the borrowing constraint is slack ($a(s) > 0$). For every s belonging to one of these intervals, equation (34) gives $\mu(s) = 0$, thus the rate of growth of consumption is the same as in Proposition 1.

We consider the simplest case in which the borrowing constraint is slack in the interval $[t, t^*)$ and binding on the interval $[t^*, T]$.[11] Obviously, t^* is endogenous and depends on the choice of the intertemporal utility functional. We denote c_t^+ as the optimal consumption profile derived from the additive model, $c_t^\times$ as the optimal consumption profile derived from the multiplicative model, and t_+^* and $t_\times^*$ the corresponding values of t^*. Thus we have:

$$\begin{cases} \forall s \in [t, t_+^*) & c_t^+(s) = c_t^+(t) \exp\left(\int_t^s \frac{r - \pi_\tau(\tau) - \beta}{\gamma} d\tau\right) \\ \forall s \in [t_+^*, T] & c_t^+(s) = w(s) \end{cases} \tag{35}$$

and

$$\begin{cases} \forall s \in [t, t_\times^*) & c_t^\times(s) = c_t^\times(t) \exp\left(\int_t^s \frac{r - \alpha\pi_\tau(\tau)}{\gamma} d\tau\right) \\ \forall s \in [t_\times^*, T] & c_t^\times(s) = w(s) \end{cases} \tag{36}$$

Of course, these two models are polar cases, and having mixed models characterized by three parameters, (α, β, γ) is possible. At some point, introducing the following notation is convenient:

$$\begin{cases} \forall s \in [t, t^*), & c_t\left[\alpha, \beta, \gamma\right](s) = c_t\left[\alpha, \beta, \gamma\right](t) \exp\left(\int_t^s \frac{r - \alpha\pi_\tau(\tau) - \beta}{\gamma} d\tau\right) \\ \forall s \in [t^*, T] & c_t\left[\alpha, \beta, \gamma\right](s) = w(s) \end{cases} \tag{37}$$

These notation results in $c_t^+ = c_t\left[1, \beta, \gamma\right]$ and $c_t^\times = c_t\left[\alpha, 0, \gamma\right]$.

The last problem to solve is the calculation of the starting level of the consumption function and the value of t^*. We start with the additive model. The *life-cycle budgetary constraint* given by equation (11) and the explicit form of the optimal consumption path given by equation (35) imply:

$$c_t^+(t) = \frac{a(t) + \int_t^{t_+^*} w(s)e^{-r(s-t)}ds}{\int_t^{t_+^*} \exp\left(\int_t^s \left(\frac{r - \pi_\tau(\tau) - \beta}{\gamma} - r\right) d\tau\right) ds} \tag{38}$$

[11] Fortunately, it is the case in all the subsequent simulations.

Because the Hamiltonian of the problem is locally strictly concave in c for every age on which w is continuous, c is continuous for all $s \neq t_R$. Thus if $t^* \neq t_R$, we also have $c(t^*) = w(t^*)$ and then :

$$c_t^+(t) = \frac{w(t_+^*)}{\exp \int_t^{t_+^*} \frac{r-\pi_\tau(\tau)-\beta}{\gamma} d\tau} \tag{39}$$

According to equations (38) and (41), t_+^* is necessarily a solution to the equation $\varphi^+(x) = 0$, with :

$$\begin{aligned}\varphi^+(x) = &\left(a(t) + \int_t^x w(s)e^{-r(s-t)}ds\right) \exp\left(\int_t^x \frac{r - \pi_\tau(\tau) - \beta}{\gamma} d\tau\right) \\ &-w(x)\int_t^x \exp\left(\int_t^s \left(\frac{r - \pi_\tau(\tau) - \beta}{\gamma} - r\right) d\tau\right) ds\end{aligned} \tag{40}$$

This equation can be solved numerically and may admit more than one solution. Leung (2007)'s theorem 3 and 4 discussed extensively the existence, uniqueness and optimality of the terminal wealth depletion time, in the case of a discontinuous wage profile. Therefore, if an isolated solution exists in the interval (t_R, T), then it is the unique optimal terminal wealth depletion time solution of the program $\mathcal{P}'_t$. That will be the case in the subsequent simulations.

The reasoning is the same for the model of the multiplicative rate of time preference:

$$c_t^\times(t) = \frac{w(t_\times^*)}{\exp \int_t^{t_\times^*} \frac{r-\alpha\pi_\tau(\tau)}{\gamma} d\tau} \tag{41}$$

and

$$\begin{aligned}\varphi^\times(x) = &\left(a(t) + \int_t^x w(s)e^{-r(s-t)}ds\right) \exp\left(\int_t^x \frac{r - \alpha\pi_\tau(\tau)}{\gamma} d\tau\right) \\ &-w(x)\int_t^x \exp\left(\int_t^s \left(\frac{r - \alpha\pi_\tau(\tau)}{\gamma} - r\right) d\tau\right) ds\end{aligned} \tag{42}$$

Finally, we choose a realistic parametrical form for the *mortality pattern* and the *life-cycle budgetary constraint*.

A law of mortality that fits the data quite well and remains tractable within an intertemporal choice model is the Gompertz law, in which the hazard rate $\pi_s(s)$ increases at a constant rate. However, using this type of law over all the lifespan necessitates allowing the maximum possible life duration T to go to infinity. Unfortunately, following this path has an important mathematical drawback, which is the difficulty in specifying a proper transversality condition with an infinite horizon. To avoid this issue, we assume a finite horizon T and that an age $T' < T$ exists such that, for all $s \in [t, T']$, $\pi_s(s) = b \exp(q(s-t))$. For $s > T'$, we will simply assumed that $\pi_s(s)$ is increasing and tends to infinity in $s = T$. As long as all of the intervals in which the borrowing constraint is not binding belong to $[t, T']$, we are able to numerically solve a parametrical version of the model, thanks to Leung (1994). In practice, we simply need to have $t^* < T'$. For the numerical simulation, we estimated the parameters b and q

using the United States Life Tables for 2009 as provided by Arias (2014). Using a simple linear fit of the log of the hazard rate between the ages of 25 and 99, we found that $b = 0.000569$ and $q = 0.0824$. We assume that $t = 25$, $T = 130$ and $T' \in (100, 130)$.

We now turn to the parametrical specification of the *life cycle budgetary constraint*. We assume that yearly net salary increases at a constant rate, g, until the age of retirement $t_R \in [t, T]$. After retirement, agents earn a constant pension. Thus, we have:

$$\begin{cases} w(s) = w(t)e^{g\,s} & \forall s \in [t, t_R) \\ w(s) = \rho\, w(t)e^{g\,t_R} = cst & \forall s \in [t_R, T] \end{cases} \tag{43}$$

with $\rho \in [0, 1]$, as the replacement ratio. This formulation of life cycle earnings introduces a discontinuity (a downward jump) in the differential constraint as in Leung (2001, 2007).

For simplicity, we normalize the initial salary at $w(t) = 1$ and choose $g = 0.005$. The age of retirement is fixed at $t_R = 65$ and the replacement ratio, $\rho = 0.60$. We also assume a long term interest rate of 3 percent per year ($r = 0.03$) and an initial asset level that corresponds to half a year's salary ($a(t) = 0.5$).

For each version of the utility function, two parameters remain to be specified: $(\alpha, \gamma^{\times})$, for the model of the *multiplicative rate of time preference* and (β, γ^{+}), for the additive one. These parameters are not directly observable, however, because they are structural, they can be inferred from the observable characteristics of the consumption hump. This hump is described by two characteristics, the age at the peak, t_H, and the size of the hump captured by the ratio $c_t(t_H)/c_t(t) = S_H$. In this section, we retain a value of $t_H = 48$ and $S_H = 1.2$.

The first step is to determine the value of the respective rate of time preference for each model. Doing this is easy considering that, at age t_H, by definition, the slope of the consumption function is equal to zero. Thus α is the solution of the one unknown linear equation $r - \alpha\pi_{t_H}(t_H) = 0$ and θ is the solution of $r - \pi_{t_H}(t_H) - \beta = 0$. Given the values of the parameters, $\pi_{t_H}(t_H) = 0.037861$. Thus, the rates of time preference corresponding to the peak of the hump at age 48 are: $\alpha = 7.92381$ and $\beta = 0.0262139$.

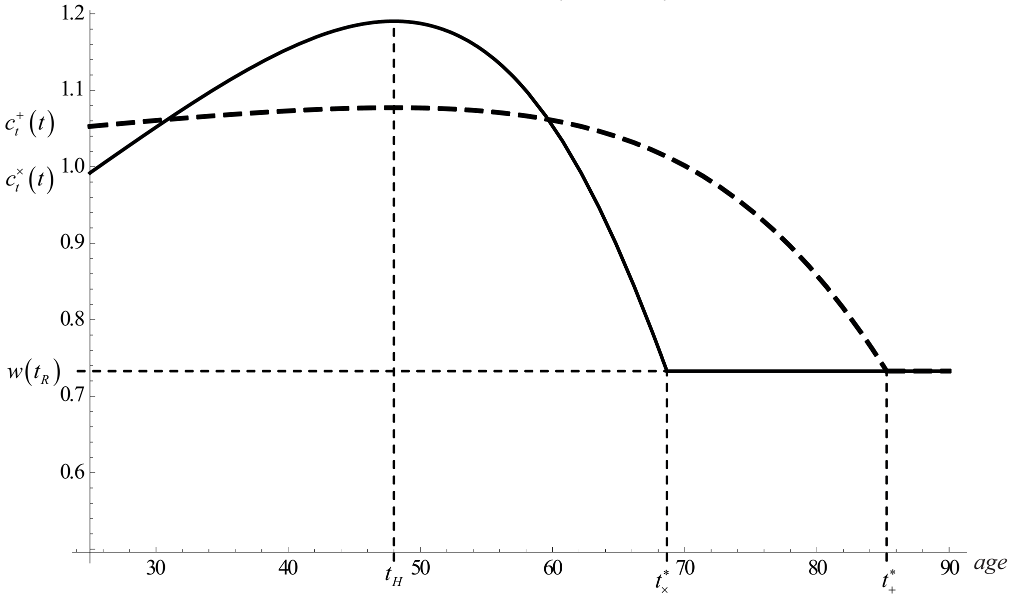
Knowing the size of the hump, we infer the value of the coefficient of relative resistance toward intertemporal substitution, γ. We start with the multiplicative model. $\gamma^{\times}$ is the solution of the following equation :

$$\int_t^{t_H} \frac{r - \alpha\pi_\tau(\tau)}{\gamma^{\times}} d\tau = \ln(S_H) \tag{44}$$

Given the parameter values, we obtain $\gamma^{\times} = 2.08773$. For a clear understanding of the compared properties of the two polar models, drawing $c_t^{\times}$ and c_t^{+} is convenient when both functions share the same coefficient of relative resistance toward intertemporal substitution (Figure 1).

By construction the two consumption profiles reach their maximum at the same age, t_H. However, as long as the borrowing constraint is slack for both profiles, $c_t^{\times}$ is everywhere steeper than c_t^{+}. This result comes from coefficient α, which is greater than 1 and magnifies the negative effect of the hazard rate. The implication is straightforward, as long as both profiles are calibrated with the same t_H and γ, we always have $c_t^{\times}(t) < c_t^{+}(t)$, $c_t^{\times}(t_H) > c_t^{+}(t - H)$ and $t_{\times}^{*} < t_{\times}^{*}$.

Figure 1: Comparison of $c_t^\times$ and c_t^+ when $\gamma^\times = \gamma^+$.



r	g	ρ	b	q	α	β	$\gamma^\times$	γ^+	$\pi_{t_H}(t_H)$
0.03	0.005	0.6	0.000569	0.0824	7.92381	0.0262139	2.08773	2.08773	0.00378606
t_H	$t_\times^*$	t_+^*	$c_t^\times(t)$	$c_t^+(t)$	$c_t^\times(t_H)$	$c_t^+(t_H)$	$w[t_R]$	$c_t^\times(t_H)/c_t^\times(t)$	$c_t^+(t_H)/c_t^+(t)$
48	68.6518	85.2763	0.991964	1.05274	1.19036	1.07725	0.732842	1.2	1.02328

Of course, to make an honest comparison, we must calibrate γ^+ to explain the same size of the hump $S_H = 1.2$ by solving the following equation:

$$\int_t^{t_H} \frac{r - \pi_\tau(\tau) - \beta}{\gamma^+} d\tau = \ln(S_H) \tag{45}$$

Given the values of parameters, we obtain $\gamma^+ = 0.263476$. Because the model of the additive rate of time preference is similar to the standard Yaari model used by Feigenbaum (2008), finding a value of γ^+ in the same order of magnitude is not surprising.[12]. The striking observation is the difference between γ^+ and $\gamma^\times$. In fact, in the model of the additive rate of time preference, for a given rate of interest, the only parameter that influences the slope of the consumption profile is precisely the relative resistance toward intertemporal substitution, γ. Therefore, a "small" γ^+ (less than 1) is required to fit the size of the hump. In contrast, as was previously noted, the multiplicative rate of time preference α magnifies the slope of the consumption profile. Therefore, a "large" $\gamma^\times$ is required to flatten the consumption profile, a very good point for the model of a multiplicative rate of time preference, because it allows a solution to the paradox of the macroeconomic literature concerning conflicting estimations on the intertemporal elasticity of substitution (i.e. the inverse of the relative rate of resistance toward intertemporal substitution). Many studies that attempted to explain the consumption hump required a γ lower than 1, whereas studies that attempted to estimate

[12] With $r = 3.5$ and $t_H = 45$, Feigenbaum (2008), p. 835, found a partial equilibrium value of $\beta = 0.027$ and $\gamma = 0.294$

the sensitivity of current consumption to variations in interest rates generally estimated a γ higher than 1 (see Attanasio and Weber, 2010, 708-711, for a survey).

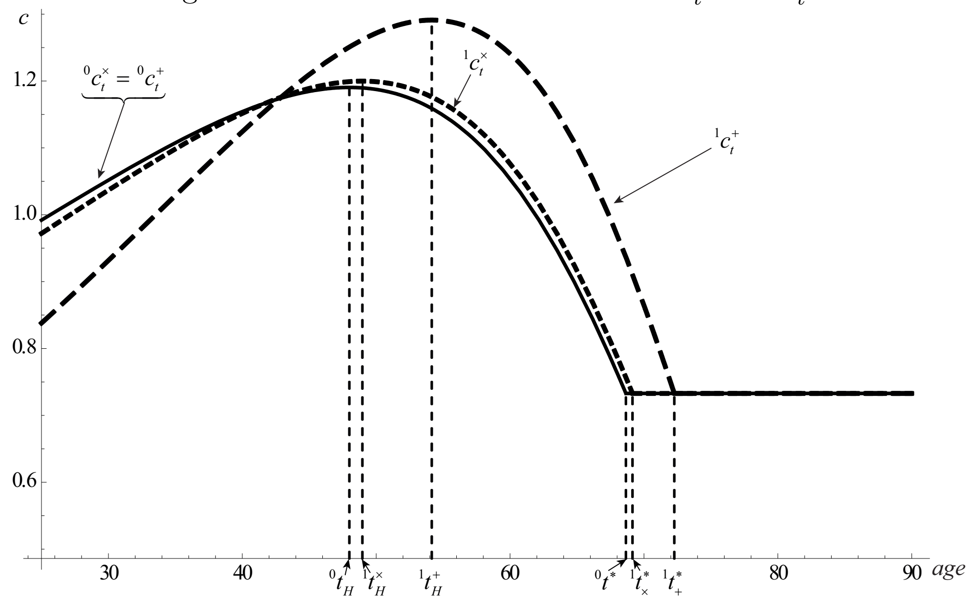
Now that we have calibrated the multiplicative model of the rate of time preference and the additive model to fit the same level of r, t_H and S_H, we answer the question on the differences between $c_t^\times$ and c_t^+. The answer is very simple, there is no difference! The linearity of the rate of growth of consumption according to the hazard rate, makes it easy to demonstrate the identity of $c_t^\times$ and c_t^+, as long as both functions are calibrated to fit the same levels of r_0, t_H and S_H and evaluated for $r = r_0$. In this case, we have:

$$\forall s \in [t, T], \qquad \frac{r_0 - \pi_s(s) - \beta}{\gamma^+} = \frac{r_0 - \alpha\pi_s(s)}{\gamma^\times} \tag{46}$$

That implies that $\gamma^+ = \gamma^\times / \alpha$ and $\beta = r_0(\alpha - 1)/\alpha$.

Of course, when coming to a comparative statics analysis of the impact of a variation in the interest rate, the two functions respond very differently. Figure 2 shows the effect of an increase in the rate of interest by 0.25 point. The left superscripts 0 and 1, for the consumption functions and endogenous variables, denote that the function/variable is calculated using, respectively, the initial and the terminal value of the interest rate. Considering the additive model of the rate of time preference, we see that

Figure 2: Effect of an increase of r on $c_t^\times$ and c_t^+.



	r_1	α	β	$\gamma^\times$	γ^+
t_H	$t_H^\times$	t_H^+	*	$t_\times^*$	t_+^*
(t)	$c_t^\times(t)$	$c_t^+(t)$	$(t_H)/\quad(t)$	$^\times(\ t_H^\times)/\quad{}^\times($	$^+(\ t_H^\times)/\quad{}^+(\ )$

a relatively small increase in the rate of interest by 0.25 point implies a very important increase of more than six years in the age at the peak of consumption ($48 \rightarrow 54.15$) and a very important increase in the size of the hump ($1.2 \rightarrow 1.54$). The additive model shows an excessive sensibility to the rate of interest, denoting the lack of robustness already pointed out by Feigenbaum (2008). This lack of robustness comes from the too low value of γ^+ required to fit the initial value of the size of the hump. In contrast, the adjustment is very smooth in the case of the multiplicative model of the rate of time preference. The age at the peak increases by less than one year ($48 \rightarrow 48.97$) and the size of the hump only increases from 1.2 to 1.23. This property of smoothness seems more realistic and implies that the multiplicative model appears to be a better candidate than the additive model for engaging in applied economic theory when intertemporal choice is involved.

5 The optimal consumption path with life annuities

As in Yaari (1965), we now assume that agents have access to actuarial notes issued by insurance companies or pension funds. Those notes are contingent assets that pay $R(s)$ as long as the agent is alive, and 0 after her death. If insurance companies refund themselves in the bond market at the rate r, and if those notes are actuarially fair, then it is well known that:

$$R(s) = r + \pi_s(s) \tag{47}$$

When no bequest motives exist, standard bonds are strictly dominated by life annuities. Thus the differential constraint of the program can be rewritten as:

$$\forall s \in [0, T], \qquad \dot{a}(s) = w(s) + R(s)a(s) - c(s) \tag{48}$$

Proceeding as in section 2, we can deduce.

$$a(t) + \int_t^T w(\tau)e^{\int_t^s -R(v)dv} = \int_t^T c(\tau)e^{\int_t^s -R(v)dv}d\tau \tag{49}$$

and

$$a(s) = a(t)\, e^{\int_t^s -R(v)dv} + \int_t^s (w(\tau) - c(\tau))\, e^{\int_\tau^s -R(v)dv}d\tau \tag{50}$$

We now deduce the property of the optimal intertemporal consumption profile when the agent has access to life annuities.

Proposition 4. *When the agent has access to life annuities, at each age s, the rate of growth of the optimal consumption path planned at date t is:*

$$\frac{\dot{c}_t(s)}{c_t(s)} = \frac{R(s) - \theta_t(s)}{\gamma_t(s)} = \frac{r + \pi_s(s) - \frac{h'(\Pi_t(s))\pi_t(s)}{1-h(\Pi_t(s))}}{\gamma_t(s)} \tag{51}$$

If the agent is time consistent and has access to life annuities, the rate of growth of the optimal consumption path planned at date t is:

$$\frac{\dot{c}_t(s)}{c_t(s)} = \frac{r + (1-\alpha)\pi_s(s) - \beta}{\gamma_t(s)} \tag{52}$$

In the polar case of the additive model of the rate of time preference *we have:*

$$\frac{\dot{c}_t^+(s)}{c_t^+(s)} = \frac{r-\beta}{\gamma_t(s)} \tag{53}$$

In the polar case of the multiplicative model of the rate of time preference*, we have:*

$$\frac{\dot{c}_t^\times(s)}{c_t^\times(s)} = \frac{r+(1-\alpha)\pi_s(s)}{\gamma_t(s)} \tag{54}$$

Proof: We proceed exactly the same way as in Proposition 1. □

The case of the *additive model of the rate of time preference* described by equation (53) corresponds precisely with the main result of Yaari (1965). In this case, when the agent has access to life annuities the rate of growth of the intertemporal consumption profile is no longer determined by the conditional probability of dying, and thus is the same as that of the model with certain life duration. For this reason, life annuities are considered as offering *perfect insurance* with the meaning that uncertainty no longer influences the rate of growth of optimal consumption. This property of Yaari's model received many comments and is perhaps at the origin of the fact that the theory of intertemporal choice of consumption and savings underestimated the role of an uncertain lefetime. For example, Barro and Friedman (1977) argued that when agent have access to life insurance/perfect annuities, the uncertainty of survival cannot be taken as a rationale for discounting.

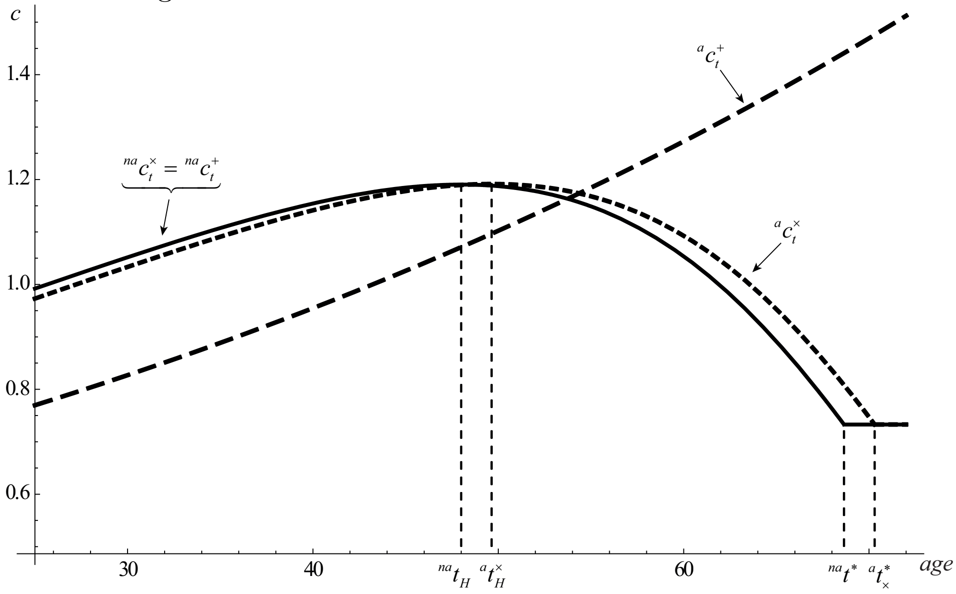
However, within the rank-dependent utility model developed in this article, the result of Yaari appears to be a very special case. As shown in Propositon 4, as long as $\alpha \neq 1$, we do not have perfect insurance, *i.e.* the conditional probability of dying still determines the rate of growth of the consumption profile. Importantly, note that the agent is fully rational in those cases and simultaneously fulfills first-order stochastic dominance and time constancy.[13]

An important consequence of the results of this section is that we can account for the consumption hump even when agents have full access to life annuities. As an example, Figure 3 considers the introduction of life annuities in the calibrated model of the preceding section and compares the effect of this introduction through the two polar models of the rate of time preference. Using the same technology as in preceding section and assuming that agents' asset cannot have a negative value after retirement, makes it is possible to draw the consumption function with or without life annuities for the two polar models. The left superscripts "a" and "na" characterize the consumption function of the agent whether she has access to life annuities or not.

If we consider the additive model of the rate of time preference, because α is equal to 1, the introduction of the life annuities and the associated risk premium exactly offset the hazard rate in the effective rate of time preference. The rate of growth of ${}^a c_t^+$ is now constant and positive over all the life cycle. The starting value of consumption is much lower (${}^{na}c_t^+(t) = 0.99, {}^a c_t^+(t) = .77$). It means that, in the additive case, the

[13]Using Selden (1978)'s *ordinal certainty equivalent* instead of *Rank-Dependent Utility*, Moresi (1999) arrived to a very similar conclusion, in the special case of an iso-elastic *per period felicity function.* The rank-dependent utility approach is much more general because it implies no restriction on the per period felicity function.

Figure 3: Effect of the introduction of life annuities



r	α	β	$\gamma^\times$	γ^+
0.03	7.92381	0.0262139	2.08773	0.263476
na	a ×	$^{na}t^*$	a $^*_\times$	a $^*_+$
48	49.6372	68.6518	70.3028	T
$^{na}c_t($ $)$	^{a} $^{\times}($ $)$	^{a} $^{+}($ $)$	$^{na}c_t($ $)/^{na}c_t($ $)$	$^{a}c_t^\times(^{a}$ $^\times)/^{a}c_t^\times(t)$
0.991964	0.972974	0.769628	1.2	1.22451

introduction of annuities has a very important effect on savings in the first half of the life-cycle. Annuities incite the agent to transfer a large mass of consumption in the second part of the life cycle. Of course having a consumption hump in the case of the model of an additive rate of time preference is impossible.

We now consider the multiplicative model of the rate of time preference. Observing the difference in the effect of the introduction of life annuities in this case is striking, and has two complementary explanations of that. First, because the calibrated level of a is relatively high (7.9381), the new slope of the consumption function is not that different from the one without life annuities. Second, the effect on consumption is comparable with the effect of an increase in the interest rate as analyzed in Figure 2. The difference is that the interest premium is no longer constant throughout the life cycle but increases with age. The age of the peak of consumption increases by approximately 1.6 years ($48 \rightarrow 49.6372$). The optimal age of capital depletion increases by approximately the same amount ($68.65 \rightarrow 70.30$).

Again the multiplicative model of the rate of time preference shows interesting smoothness properties, and is characterized by a hump in consumption that is impossible to obtain within the expected utility framework with an additively separable life-cycle utility function. By dropping the additive separability of the intertemporal utility functional, Bommier (2013) demonstrated for the first time the possibility of a hump in consumption when the agent has full access to life annuities. In this article, we showed that an other strategy is possible. By keeping an additively separable life cycle utility functional, but using a more general rank-dependent utility framework for dealing with an uncertain lifetime, we also demonstrated the possibility of accounting for the consumption hump.[14] In both cases an increase in risk aversion introduced by the model and demonstrated by (Bommier et al., 2012) accounts for the possibility of the hump.

6 Conclusion

Generally, conventional wisdom considers exponential discounting and expected utility as the only models of choice that are compatible with full rationality. If these models do not fit agents' actual behavior, then considering alternative descriptive/behavioral models is legitimate. Believing in the conventional wisdom allows for the deduction that agents are not rational. Obviously, from a normative point of view, this discussion has very important implications for policy design.

In this article we built a model of intertemporal choice that uses the rank-dependent utility theory to deal with intertemporal choice when life duration is uncertain. We showed, that a subclass of this model is compatible with full rationality, when agents transform the decumulative distribution of probability using a power function. The power coefficient of the transformation is then interpreted as a multiplicative rate of time preference. The model has the same level of complexity as the traditional expected utility model originally designed by Yaari (1965) and allows for simple and intuitive interpretations. Using a simple calibration, we showed that this model can be an interesting candidate for solving some of the puzzles from the empirical literature (for example, hump-shaped life-cycle consumption function and a too low coefficient of relative intertemporal resistance). We hope that the example drawn in this article will attract the attention of theoretical and applied economists and incite them to use it as an alternative to the standard expected utility/exponential discounting model.

[14]Building on Bommier (2013), Bommier and Le Grand (2014) consider an altruistic bequest motive and shows that the demand for annuity is a decreasing function of the lifetime risk aversion. An extension of the RDU model of this study incorporating such a bequest motive will probably emphasize the role of α in determining the optimal shares of bond and annuities in the agent's portfolio.

Appendix A - The Allais paradox and Rank-Dependent Utility.

The Allais Paradox (Allais, 1953) is an experimental refutation of expected utility for representing agents' preferences under risk. The agent chooses between "lotteries" that are characterized by a vector of "monetary gains" and the vector of associated probabilities.

In the first part of the experiment, the agent chooses between two lotteries $A_1\{1\$m; 1\}$ and $B_1\{(0, 1\$m, 5\$m); (0.01, 0.89, 0.1)\}$. In the second part of the experiment she chooses between $A_2\{(0, 1\$m); (0.89, 0.11)\}$ and $B_2\{(0, 5\$m); (0.9, 0.1)\}$. When confronted with this experimental choice, a large majority of subjects choose $A_1 \succ B_1$ and $B_2 \succ A_2$, that is a violation of expected utility theory. The easiest way to understand why, is to use a graphical tool, the "Marschak-Machina" triangle (Machina, 1983). This triangle is used to represent all the lotteries based on a given set of three ordered outcomes, here (0,1\$m,5\$m). The horizontal axis represents the probability, p_1, of the "lowest" gain, the vertical axis represents the probability, p_3, of the "highest gain". Since $p_2 = 1 - p_1 - p_3$, each point belonging to the unity triangle represent a possible lottery.

Indifference curves can also be drawn in this triangle. If the agent is an expected utility maximizers, then it is easy to show that, because the Expected Utility model is linear in probabilities, the indifference curves of the agent are parallel straight lines.

Figure 4-a represents the lotteries of the Allais Paradox in the triangle and an example of indifference curves of an expected utility maximizer agent. By construction, $\overrightarrow{A_1B_1} = \overrightarrow{A_2B_2}$. If the agent prefers A_1 to B_1 then the parallel indifference lines have to be steeper than (A_1B_1). If the agent prefers B_2 to A_2, then the parallel indifference lines have to be flatter than (A_2B_2). It is of course impossible to have both.

This graphical representation of the paradox enables to understand that to solve the paradox requires to drop the linearity in probability of the expected utility representation. It means that agents not only subjectively transform the outcomes with a utility function, but also, by a mean or another, the probabilities. Finding the right way to model how probabilities are transformed has been a long quest. In particular, a direct transformation of the probability of each particular outcome implies that decision weights no longer sum to one, and that violation of *first-order stochastic dominance* is possible. Quiggin (1982)'s *anticipated utility model* solved the problem for the first time, with what is today commonly known as the *rank-dependent utility model* (Quiggin, 1993). The idea his simple, the agents have to transform the whole probability distribution. Therefore, the weighting function should apply to the cumulative distribution function.

When gains are properly ranked ($x_1 < x_2 < ... < x_n$) the value of the lottery is given by:

$$RDU(X, P) = \sum_{i=1}^{n} h_i(P)u(x_i)$$

with $h_i(P) = w\left(\sum_{j=1}^{i} p_j\right) - w\left(\sum_{j=1}^{i-1} p_j\right)$ and $w : [0, 1] \to [0, 1]$ with $h(0) = 0$, $h(1) = 1$ and $h' > 0$.

It can easily be shown, that, when the probability weighting function is concave, then the indifference curves in the Marschak-Machina triangle will also be concave. Figure 4-a shows an example of a solution of the Allais Paradox when the weighting function is concave. This example is important here, because this study precisely emphasizes a special case of a concave transformation of the cumulative probability function.

Rank-dependent utility can also be used for continuous random variables. Let x be a continuous random variable defined on X, with F the cumulative distribution function assumed to be C^1. Then the rank-dependent model can be written :

$$RDU(X, F) = \int_X x \, dh$$

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