Value-at-Risk-Based Risk Management in A Jump-Diffusion Model

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March 30, 2013

Abstract

This paper investigates optimal portfolio and wealth strategy of an institutional investor with a Value-at-Risk (VaR) constraint in an economy under jump diffusion. The problem is solved in closed form. We show that overlooking or underestimating the jump risk factor results in failures to satisfy the VaR constraint. We also find that the presence of the jump risk drives the institutional investor to move towards the portfolio insurance strategy, alleviating the problem with VaR identified by Basak and Shapiro (2001) that the VaR risk manager incurs larger losses than non risk manager in worst scenarios. Moreover, put options play an important role in hedging jump risks under the VaR framework.

Keywords: Asset Allocation, Options, Value-at-Risk, Risk Management
JEL Codes: G11; G12

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1 Introduction

The past few years have witnessed a global financial meltdown, which raised a heated discussion on the massive failures of risk measurement and management in the financial industry. As realized by academics and practitioners, both ignoring known risk and devising the wrong response to risk can lead to financial mismanagement (see, for example, Stulz (2009)). This implies that the risk mismanagement in the recent financial crisis is likely to stem from overlooking some major risk factors such as jump risk and using the risk measures in an inappropriate way. On the other hand, as an easily understandable and interpretable measure of risk, VaR has been extensively used by financial as well as nonfinancial firms and become established as the industry and regulatory standard in measuring market risk over the past 15 years. Despite its wide popularity, VaR has continually come up short, particularly in recent financial crisis. A typical example can be observed in the violent market upheavals of 2007-2008, where many banks reported more than 30 days when losses exceeded VAR, while 3 to 5 such days would be the norm. In particular, one major limitation of VaR is its reliance on normal market distributions; it ignores the empirically observed "skewed" and "fat-tailed" features of stock return distributions. As jumps in stock returns can generate such features, it is of interest to incorporate the jump risk into account the VaR risk management framework and see how it affects the optimal portfolio strategies of institutional investors.

In this paper, we examine the optimal portfolio and wealth strategies for an institutional investor maximizing her utility and facing a Value-at-Risk (VaR) constraint at the investment horizon in a jump-diffusion model. In particular, we study the implications of adding jump risk to the VaR risk management for optimal wealth choice and portfolio policies. To complete the market, the investor is given the access not only to the bond and stock markets but also to the options market to manage the exposure to the diffusive risk factor and jump risk factor. We also investigate the role of put options in hedging jump risk under the VaR framework.

This article is related to the strand of literature on the dynamic asset allocation problem under risk measure constraints in a variety of settings. Basak and Shapiro (2001) study the

institutional investors' portfolio composition and wealth polices subject to a single-VaR constraint and a single Limited-Expected-Losses (LEL) constraint and find that the presence of the VaR constraint induces the agent to take larger equity in worst scenarios and the LEL constraint turns out to remedy this shortcoming. By allowing for dynamically reevaluated VaR limit, Cuoco, He, and Isaenko (2008) find that a VaR limit does not necessarily generate negative impact and actually serves as an appropriate tool in managing risk. Shi and Werker (2012) extend Cuoco, He, and Isaenko (2008) by taking into the interest rate risk and allowing for rebalancing of the portfolio over the regulatory horizon. All of these papers assume that stock prices follow pure diffusion processes. However, many empirical studies have documented evidence of jumps in stock returns; see, for example, Bakshi, Cao, and Chen (1997) and Eraker, Johannes, and Polson (2003). With jumps, stock market returns display negative skewness and fat left tail. A vast literature has documented that the jump risk has substantial influence on portfolio choice and risk management; see, Duffie, Pan, and Singleton (2000) and Liu, Longstaff, and Pan (2003), Liu and Pan (2003), for example. To this end, we extend Basak and Shapiro (2001) by adding jump risk and making stock return distribution positively skewed. To the best of our knowledge, our paper is the first attempt to study the effect of jump risk on the optimal portfolio strategy in the VaR risk management framework.

This paper also builds on the literature on options investment and pricing. Liu and Pan (2003) solve the portfolio choice problem with both jump risk and stochastic volatility and find sizable portfolio improvement from derivatives investment. Tan (2009) analyzes the attractiveness of European style call and put options for long horizon investors. Other papers include Zhang, Zhao, and Chang (2012) and Kou (2002), which mainly focus on market incompleteness and option pricing. In addition to theoretical studies, a variety of empirical works estimate the jump risk premium embedded in the options (see, for example, Bates (2000) and Pan (2002)). In principle, most papers focus on the premium delivered by options and emphasize their speculative role, although it is well known that put options are capable of hedging downside risk. Surprisingly, the academic literature has largely overlooked the hedging role of options. As an exception, Ahn, Boudoukh, Richardson, and Whitelaw (1999) analyzes how

useful the options are in minimizing the firms' VaR. They assume a standard Black Scholes world in which the options are redundant, and focus on how to optimally choose the strike price. In contrast, we intend to examine asset allocation problem by employing options to manage the jump risk in a VaR framework and highlight the hedging role of options.

Our main results are as follows: First, consistent with Basak and Shapiro (2001), we find that an agent with a VaR limit optimally chooses to insure against intermediate states, while incurring losses in the worst states, the probability under which is exactly the prespecified level α. However, we show that ignoring jump risk leads to failures of risk management, because the institutional investor makes too aggressive investment decisions. The mistakes can be attributed to the fat upper tail of the distribution of pricing kernel generated by the jump risk component; if the agent overlooks jump risk and adopts the wealth policy proposed by Basak and Shapiro (2001), she fails to satisfy the VaR constraint, because the fat upper tail makes the probability under the uninsured bad states exceed α.

Second, the presence of jump risk induces the VaR agent to maintain higher wealth in the intermediate and worst states and move towards the portfolio insurance strategy, thereby mitigating the problem with VaR identified by Basak and Shapiro (2001) that the VaR agent optimally takes larger exposure to risky assets in unfavorable states and incurs larger losses than the benchmark agent without the VaR constraint. Put differently, the VaR agent invests more conservatively when taking into account jump risk and this effect is more pronounced for higher jump risk premium.

Third, the intuitional investor employs the put option for both purposes of hedging and speculation. Compared with Liu and Pan (2003), our results substantiate the hedging role from several aspects. First, when the pricing kernel does not have jumps, the investor invests more of her wealth in the option in our VaR framework than in Liu and Pan (2003), which is demonstrated by both higher option holding and lower stock-option ratio. Second, when jumps become much rare and large, the agent is reluctant to get large exposure to jump risk, even though there is a high risk premium. This observation follows from the fact that the agent needs to prepare for those worst scenarios to satisfy the VaR constraint. Finally, in general,

our stock-option ratio is lower than that of Liu and Pan (2003), indicating stronger hedging demand for options. This distinction can be explained by the effect of the VaR constraint; with the VaR constraint, the agent has to care more about jump risk and therefore holds more put options.

The contributions of this paper are twofold. First, we extend the pure diffusion model of Basak and Shapiro (2001) to a jump-diffusion model and investigate the impact of the jump risk on the risk management strategy under the VaR framework. Our results may shed some light on the cause of risk management failures in the recent financial crisis. A rational explanation offered by our model is that if the VaR risk managers rely on normal market conditions and ignore jump risk, they underestimate the probability of their portfolio falling below the lower threshold and invest too aggressively. As a consequence, huge losses may arise when the economic conditions deteriorate and stock price jumps are realized. Second, we find higher allocation to put options when investors are confronted with the VaR constraint, which is likely to be driven by stronger hedging demand. A challenging puzzle regarding index puts is where is the source of demand; Driessen and Maenhout (2007) document that long positions seem anomalously suboptimal in the portfolio choice problem. The finding that the higher put option demand is associated with stronger portfolio insurance purposes under the VaR constraint sheds some light on this puzzle.

The rest of the paper is organized as follows. Section 2 describes the model. Section 3 investigates the optimal wealth and portfolio strategy under the VaR constraint. Section 4 concludes the paper.

2   The Model

We consider a complete financial market with a finite horizon $[0, T]$. In this market, three securities are available: a riskless bond paying a constant interest rate r, a risky stock representing the aggregate equity market and an option based on the stock. Denote by B, S, O the price processes of the bond, the stock and the option. The dynamics for the price processes of

the bond and the stock are assumed as follows:

$$\frac{\mathrm{d}B_t}{B_{t-}} = r\mathrm{d}t, \tag{1}$$

$$\frac{\mathrm{d}S_t}{S_{t-}} = \left(r + \eta\sigma + \mu\left(\lambda - \lambda^Q\right)\right)\mathrm{d}t + \sigma\mathrm{d}Z_t + \mu\left(\mathrm{d}N_t - \lambda\mathrm{d}t\right), \tag{2}$$

where Z is a standard Brownian motion and N is a pure-jump process. They are assumed to be independent of each other. σ is the equity market volatility. μ and λ are the jump size and jump arrival intensity associated with the pure-jump process N. Following Liu and Pan (2003), we assume constant jump amplitudes, which imply that conditional on a jump arrival, the stock return jumps by a constant $-1 < \mu < 0$, with the limiting case of -1 capturing total loss. This specification of deterministic jump sizes simplifies the analysis, as only one extra derivative security is needed to complete the market with respect to the jump risk component. Although stochastic volatility and jump in volatility are common in the literature on options (see, for example, Liu and Pan (2003) and Branger, Schlag, and Schneider (2008)), we do not incorporate these elements in our model, because our focus is not on the empirical properties of options but rather on the the role of options in a VaR risk management framework. A noteworthy feature of (2) is that the stock return is no longer normally distributed, because the presence of negative jump risk produces positive skewness. Therefore, the inclusion of jump component allows us to relax the assumption of normal market condition, the reliance on which is regarded as one major limitation of the VaR risk management framework, and generate large tail risk, which is consistent with the empirically observed stock return distribution. Finally, η and λ^Q capture the two components of the equity premium: one is for diffusive risk Z and the other for jump risk N.

In order to price the option, we first specify the pricing kernel of the economy. The complete market assumption implies, there exits a unique pricing kernel ξ, whose dynamics is,

$$\frac{\mathrm{d}\xi_t}{\xi_{t-}} = -r\mathrm{d}t - \eta\mathrm{d}Z_t - \left(1 - \frac{\lambda^Q}{\lambda}\right)\left(\mathrm{d}N_t - \lambda\mathrm{d}t\right) \tag{3}$$

with $\xi_0 = 1$ and η and $(1 - \lambda^Q/\lambda)$ are the market price of diffusive risk and that of jump risk,

respectively. As we focus on negative jumps $\mu < 0$, the market price of jump risk must be negative, which mandates $\lambda^Q > \lambda$. As become clear later, the introduction of jump risk also makes the distribution of the pricing kernel more positively skewed, which, has substantial impact on the risk management and measurement. It is important to note that the jump risk premium determines the jump amplitude of ξ_t. When there is no jump risk premium ($\lambda^Q = \lambda$), the third term in (3) drops out and (3) reduces to the pricing kernel in the absence of jump risk. Consistent with this pricing kernel is the following option price dynamics:

$$\frac{\mathrm{d}O_t}{O_{t-}} = \left(r + \eta \frac{O_S}{O_t} S_t \sigma + \frac{\Delta O}{O_t} \left(\lambda - \lambda^Q \right) \right) \mathrm{d}t + \frac{O_S S_t}{O_{t-}} \sigma \mathrm{d}Z_t + \frac{\Delta O}{\mu O_{t-}} \mu \left(\mathrm{d}N_t - \lambda \mathrm{d}t \right) \quad (4)$$

where O_S and ΔO are given by,

$$O_S = \frac{\partial O(S)}{\partial S}; \qquad \Delta O = O((1+\mu)S_t) - O(S_t). \quad (5)$$

Specifically, O_S measures the sensitivity of the option price to the infinitesimal changes in the underlying stock price and ΔO measures the sensitivity to the large changes in the stock price. Note that complete market assumption ensures that both O_S and ΔO can be analytically determined. To complete the market with respect to jump risk, the option price must have different sensitivities to infinitesimal and large changes in stock prices: $O_S \neq \frac{\Delta O}{\mu S_t}$.

The self-financing condition implies that the wealth process evolves as,

$$\frac{\mathrm{d}W_t}{W_{t-}} = \left(r + \pi_t^Z \eta \sigma + \pi_{t-}^N \mu \left(\lambda - \lambda^Q \right) \right) \mathrm{d}t + \pi_t^Z \sigma \mathrm{d}Z_t + \pi_{t-}^N \mu (\mathrm{d}N_t - \lambda \mathrm{d}t), \quad (6)$$

where π_t^Z and π_t^N capture the exposure of the wealth process to diffusive risk and jump risk respectively. Denote by x^S and x^O the fractions of wealth invested in the stock and the option. Then, π_t^Z and π_t^N are defined by,

$$\pi_t^Z = x_t^S + x_t^O \frac{O_S S_t}{O_t}, \quad (7)$$

$$\pi_t^N = x_t^S + x_t^O \frac{\Delta O}{\mu O_t}. \quad (8)$$

(7) and (8) imply that investing x^S of the wealth in the stock and x^O in the derivative security amounts to investing π_t^Z in the diffusive risk factor Z and π_t^N in the jump risk factor N. It is worth emphasizing that the maturity of the option does not have to match the investment horizon, as rebalancing of the portfolio is allowed at each point in time in the future.

3 Optimization under the VaR constraint

3.1 Optimal Portfolio Wealth

We consider an institutional investor, who is initially endowed with wealth of W_0 and is concerned with maximizing the expected utility over the terminal wealth. We assume that the institutional investor has CRRA preferences with relative risk aversion of γ and a fixed investment horizon of T.

The institutional investor is subject to a constraint of a VaR type at the horizon imposed by the regulator, which can be formulated as

$$P(W_T \leq \underline{W}) \leq \alpha, \tag{9}$$

where the "floor" $\underline{W}$ and the loss probability α are exogenously specified. The VaR constraint requires that the probability that the institutional investor's wealth at the horizon falls below the floor wealth $\underline{W}$ be α or less. Following Basak and Shapiro (2001), We also consider two alternative cases: one is the benchmark case (B), in which the VaR constraint is never binding and the other one is the portfolio insurance case (PI), in which the horizon wealth is constrained to be above the floor $\underline{W}$ in all states. Note that (9) nests both the B-case and the PI-case, which correspond to $\alpha = 1$ and $\alpha = 0$, respectively. Therefore, the VaR-case can be thought of as an intermediate case between the two extreme cases, the B-case and the PI-case.

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The portfolio optimization problem of the VaR agent can be formulated as follows,

$$\max_{W_T} \mathrm{E}\left[\frac{W_T^{1-\gamma}}{1-\gamma}\right] \tag{10}$$

$$\text{subject to} \quad \mathrm{E}[\xi_T W_T] \leqslant \xi_0 W_0 \tag{11}$$

$$P(W_T \leqslant \underline{W}) \leqslant \alpha. \tag{12}$$

Following Basak and Shapiro (2001), we solve this problem using the martingale representation approach. Proposition 1 characterizes the optimal terminal wealth under the VaR constraint.

Proposition 1. *The time-T optimal wealth of the VaR agent is*

$$W_T^{VaR} = \begin{cases} \left(y^{VaR}\xi_T\right)^{-\frac{1}{\gamma}} & \text{if } \xi_T < \underline{\xi}^{VaR}, \\ \underline{W} & \text{if } \underline{\xi}^{VaR} \leqslant \xi_T < \bar{\xi}^{VaR}, \\ \left(y^{VaR}\xi_T\right)^{-\frac{1}{\gamma}} & \text{if } \xi_T \geqslant \bar{\xi}^{VaR}. \end{cases} \tag{13}$$

where $\underline{\xi}^{VaR} = \frac{\underline{W}^{-\gamma}}{y^{VaR}}$, $\bar{\xi}^{VaR}$ is such that $P(\xi_T > \bar{\xi}^{VaR}) = \alpha$, and y^{VaR} is the Lagrange multiplier of the budget constraint and solves $\mathrm{E}\left[\xi_T W_T^{VaR}\right] = \xi_0 W_0$. The VaR constraint is binding if and only if $\underline{\xi}^{VaR} < \bar{\xi}^{VaR}$. Moreover, $y^{VaR} \in \left[y^B, y^{PI}\right]$.

Proposition 1 indicates that if the VaR constraint is binding, the VaR agent's optimal horizon wealth is classified into three distinct regions: in both regions of "good states" $\left[\xi_T < \underline{\xi}^{VaR}\right]$ and "bad states" $\left[\xi_T \geqslant \bar{\xi}^{VaR}\right]$, her terminal wealth is decreasing in ξ_T, while in the region of "intermediate states" $\left[\underline{\xi}^{VaR} \leqslant \xi_T < \bar{\xi}^{VaR}\right]$, her terminal wealth is kept constant at the portfolio insurance level. The definition of the upper bound $\bar{\xi}^{VaR}$ implies that the probability under the bad states region stays constant at α. Moreover, $y^{VaR} \in \left[y^B, y^{PI}\right]$ confirms that the VaR case is intermediate between the B case and the PI case. Note that Proposition 1 coincides with the Basak and Shapiro (2001) solution except for the specific CRRA utility function. This follows from the fact that optimal wealth policy under the VaR constraint is fully characterized in terms of the pricing kernel, while jump risk affects the optimal terminal wealth strategy solely

through its impact the distribution of the pricing kernel. The distinction between the optimal wealth policies in the two cases will be made clear later.

Basak and Shapiro (2001) show that the VaR agent's optimal terminal wealth in (13) can be decomposed as

$$
\begin{aligned}
W_T^{VaR}(y(W_0)) &= W_T^{PI}(y^B(W_*)) - (\underline{W} - W_T^B(y^B(W_*)))1_{\{\xi_T \geqslant \bar{\xi}\}} && (14) \\
&= W_T^B(y^B(W_*)) + (\underline{W} - W_T^B(y^B(W_*)))1_{\{\underline{\xi} \leqslant \xi_T < \bar{\xi}\}}, && (15)
\end{aligned}
$$

where W_* is set so that $y^B(W_*) = y(W_0)$. Put differently, W^{VaR} is equivalent to a PI solution plus a short position in "binary" options, or a B solution plus an appropriate position in "corridor" options.

[Insert Figure 1 about here.]

Figure 1 illustrates the optimal terminal wealth of a VaR agent, a benchmark agent $(\alpha = 1)$ and a portfolio insurance agent $(\alpha = 0)$. Consistent with Proposition 1, in both regions of good states and bad states, the VaR agent behaves like the B agent. In contrast, in the intermediate states she adopts portfolio insurance strategy as the PI agent does. A striking feature of the VaR agent's horizon wealth strategy is that she leaves the bad states fully uninsured, as they are most costly to insure against; her wealth is even lower than the B agent's wealth in the worst state for any given ξ_T. In other words, the VaR agent ignores losses in the upper tail of the ξ_T distribution, which is independent of the agent's preferences and endowment but dependent on the jump size of the pricing kernel. This worse performance of the VaR agent unveils a shortcoming of VaR that it creates incentive to take on tail risk.

[Insert Figure 2 about here.]

Figure 2 illustrates the distribution of the pricing kernel at horizon for different jump parameters. Compared with the Basak and Shapiro case, the introduction of jump risk makes the distribution of ξ_T more positively skewed and the upper tail of the distribution fatter. This effect is more pronounced for higher jump risk premium, which implies larger jump size. The

detrimental consequence of these variations is that the probability beyond the upper bound of the states $\bar{\xi}^{VaR}$ implied by Basak and Shapiro (2001) exceeds the pre-specified loss probability α, thereby leading to violation of the VaR constraints. This observation is more clearly confirmed by the right panel, which plots the upper tails of the distribution of ξ_T in different cases. In other words, if an institutional investor follows the terminal wealth distribution proposed by Basak and Shapiro (2001) in a market with positive jump risk premium, she will mistakenly choose a larger region as bad states than what is implied by the distribution of ξ_T and make too aggressive investment decisions. Therefore, overlooking jump risk could be a cause of failure to satisfy the VaR constraints in the recent financial crisis for many institutional investors.

[Insert Table 1 about here]

Table 1 reports the classification of states for optimal horizon wealth under the VaR constraint for different jump parameters. the result in the no jump component case coincides with that in the no jump risk premium case ($\lambda = \lambda^Q = 1$). This follows from the fact that the redistribution of the terminal wealth driven by the VaR constraint is completely determined by the properties of the pricing kernel. In contrast, all of the upper bounds associated with positive jump risk premium exceed the upper bound in the Basak and Shapiro case, which justifies underestimating jump risk as an alternative explanation for the failures of risk management in the recent financial crisis. The upper bound $\bar{\xi}^{VaR}$ is decreasing in λ for any given λ^Q, but increasing in λ^Q for any given λ. In contrast, the opposite holds for the lower bound $\underline{\xi}^{VaR}$. As a consequence, the region of the intermediate states widens as the jump risk premium increases. These results are obviously consistent with Figure 1: as the tail gets fat, $\bar{\xi}^{VaR}$ must increase to make sure that the probability beyond $\bar{\xi}^{VaR}$ is equal to α. On the other hand, $\underline{\xi}^{VaR}$ must decrease to satisfy the budget constraint due to the more positively skewed distribution of ξ_T. While the probability in the region of worst states remains constant at α as prescribed by the VaR constraint, the probability in each of the two regions varies across different jump risk premiums. Since the VaR agent chooses to fully insure against the intermediate states, one can think of the probability in the region of the intermediate states as the cost of satisfying the VaR constraint. In Table 1, we denote the probability that the terminal wealth is under portfolio

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insurance for the VaR agent by Pr^{VaR} and that for the PI agent by Pr^{PI}. It is revealed that in general, both Pr^{VaR} and Pr^{PI} decrease with the jump risk premium, indicating lower costs of meeting the VaR constraints. This can be explained by the fact that although $\underline{\xi}^{VaR}$ shifts to the left, the distribution of ξ_T becomes more positively skewed with the jump risk premium, rendering the probability under portfolio insurance to shrink. In addition, while the Lagrange multiplier in the B case keeps constant, the Lagrange multiplier in both the VaR case and the PI case increases with the jump risk premium, implying that the introduction of jump risk premium induces both the VaR and the PI agents to deviate more from the benchmark case.

[Insert Table 2 about here]

To quantify the economic significance of jump risk under the VaR framework, we compute the certainty equivalent wealth in a variety of cases. The initial wealth is assumed to be one and the investment horizon is one year. Inspection of the results in Table 2 establishes that the certainty equivalent wealth in the absence of jump risk is lower than that in the presence jump risk and it increases with jump risk premium (λ_Q/λ). This is consistent with the previous observation that jump risk makes the VaR constraint stronger and more difficult to meet and therefore results in higher utility losses. On the other hand, the certainty equivalent wealth decreases with risk aversion, implying that the utility losses associated with the VaR constraint are higher for less risk-averse investors.

3.2   Trading Strategies

Proposition 2 characterizes the optimal wealth and portfolio strategies before the horizon under the VaR constraint.

Proposition 2. *The time-t optimal wealth is given by*

$$W_t^{VaR} = \frac{\mathrm{e}^{\Gamma t}}{(y\xi_t)^{\frac{1}{\gamma}}} - \left\{ \frac{\mathrm{e}^{\Gamma t}}{(y\xi_t)^{\frac{1}{\gamma}}} \mathrm{E}_t \left[\mathrm{e}^{\frac{1-\gamma}{\gamma}\Psi(N_t)} \mathcal{N}\left(-d_1(\underline{\xi}^{VaR}) - \frac{\Psi(N_t)}{\eta\sqrt{\tau}} \right) \right] \right.$$

$$- \underline{W}\mathrm{e}^{-r\tau}\mathrm{E}_t \left[\mathrm{e}^{-\Psi(N_t)} \mathcal{N}\left(-d_2(\underline{\xi}^{VaR}) - \frac{\Psi(N_t)}{\eta\sqrt{\tau}} \right) \right] \right\}$$

$$+ \left\{ \frac{\mathrm{e}^{\Gamma t}}{(y\xi_t)^{\frac{1}{\gamma}}} \mathrm{E}_t \left[\mathrm{e}^{\frac{1-\gamma}{\gamma}\Psi(N_t)} \mathcal{N}\left(-d_1(\bar{\xi}^{VaR}) - \frac{\Psi(N_t)}{\eta\sqrt{\tau}} \right) \right] \right.$$

$$\left. - \underline{W}\mathrm{e}^{-r\tau}\mathrm{E}_t \left[\mathrm{e}^{-\Psi(N_t)} \mathcal{N}\left(-d_2(\bar{\xi}^{VaR}) - \frac{\Psi(N_t)}{\eta\sqrt{\tau}} \right) \right] \right\} \tag{16}$$

where $\tau = T - t$, y is given in Proposition 1, $\mathcal{N}(\cdot)$ is the standard-normal cumulative distribution function and

$$\underline{\xi}^{VaR} = \frac{1}{y\underline{W}^\gamma},$$

$$\Gamma(t) = \frac{1-\gamma}{\gamma}\left(r + \frac{\eta^2}{2} \right)\tau + \left(\frac{1-\gamma}{\gamma} \right)^2 \frac{\eta^2}{2}\tau,$$

$$\Psi(N_t) = (\lambda^Q - \lambda)\tau - \ln\left(\frac{\lambda^Q}{\lambda} \right)(N_T - N_t),$$

$$d_2(x) = \frac{\ln\left(\frac{x}{\xi_t} \right) + \left(r - \frac{\eta^2}{2} \right)\tau}{\eta\sqrt{\tau}},$$

$$d_1(x) = d_2(x) + \frac{\eta}{\gamma}\sqrt{\tau}.$$

Note that all of the expectations in (16) are taken with respect to $(N_T - N_t)$.

The exposure of the optimal portfolio to the risk factors Z and N is given by,

$$\pi_t^{Z,VaR} = \frac{\eta}{\sigma\gamma} - \frac{\mathrm{e}^{-r\tau}\eta\underline{W}}{\sigma\gamma W_t^{VaR}}\mathrm{E}_t\left[\mathrm{e}^{-\Psi(N_t)}\left(\mathcal{N}\left(-d_2(\underline{\xi}^{VaR}) - \frac{\Psi(N_t)}{\eta\sqrt{\tau}}\right) - \mathcal{N}\left(-d_2(\bar{\xi}^{VaR}) - \frac{\Psi(N_t)}{\eta\sqrt{\tau}}\right)\right)\right]$$
$$+ \frac{\mathrm{e}^{-r\tau}\underline{W}}{\sigma\sqrt{\tau}W_t^{VaR}}\mathrm{E}_t\left[\mathrm{e}^{-\Psi(N_t)}\left(\phi\left(d_2(\bar{\xi}^{VaR}) + \frac{\Psi(N_t)}{\eta\sqrt{\tau}}\right) - \phi\left(d_2(\underline{\xi}^{VaR}) + \frac{\Psi(N_t)}{\eta\sqrt{\tau}}\right)\right)\right]$$
$$- \frac{\mathrm{e}^{\Gamma_t}(y\xi_t)^{-\frac{1}{\gamma}}}{\sigma\sqrt{\tau}W_t^{VaR}}\mathrm{E}_t\left[\mathrm{e}^{\frac{1-\gamma}{\gamma}\Psi(N_t)}\left(\phi\left(d_1(\bar{\xi}^{VaR}) + \frac{\Psi(N_t)}{\eta\sqrt{\tau}}\right) - \phi\left(d_1(\underline{\xi}^{VaR}) + \frac{\Psi(N_t)}{\eta\sqrt{\tau}}\right)\right)\right],$$

$$\tag{17}$$

$$\pi_t^{N,VaR} = \frac{\sigma\left(1 - \frac{\lambda^Q}{\lambda}\right)}{\mu\eta}\pi_t^{Z,VaR}. \tag{18}$$

In the benchmark case, the exposure of the optimal portfolio to the risk factor Z is,

$$\pi_t^B = \frac{\eta}{\gamma\sigma} \tag{19}$$

Let $q_t^{VaR} = \pi_t^{Z,VaR}/\pi_t^B$. Then q^{VaR} is,

$$q_t^{VaR} = 1 - \frac{\mathrm{e}^{-r\tau}\underline{W}}{W_t^{VaR}}\mathrm{E}_t\left[\mathrm{e}^{-\Psi(N_t)}\left(\mathcal{N}\left(-d_2(\underline{\xi}^{VaR}) - \frac{\Psi(N_t)}{\eta\sqrt{\tau}}\right) - \mathcal{N}\left(-d_2(\bar{\xi}^{VaR}) - \frac{\Psi(N_t)}{\eta\sqrt{\tau}}\right)\right)\right]$$
$$+ \frac{\gamma\mathrm{e}^{-r\tau}\underline{W}}{\eta\sqrt{\tau}W_t^{VaR}}\mathrm{E}_t\left[\mathrm{e}^{-\Psi(N_t)}\left(\phi\left(d_2(\bar{\xi}^{VaR}) + \frac{\Psi(N_t)}{\eta\sqrt{\tau}}\right) - \phi\left(d_2(\underline{\xi}^{VaR}) + \frac{\Psi(N_t)}{\eta\sqrt{\tau}}\right)\right)\right]$$
$$- \frac{\gamma\mathrm{e}^{\Gamma_t}(y\xi_t)^{-\frac{1}{\gamma}}}{\eta\sqrt{\tau}W_t^{VaR}}\mathrm{E}_t\left[\mathrm{e}^{\frac{1-\gamma}{\gamma}\Psi(N_t)}\left(\phi\left(d_1(\bar{\xi}^{VaR}) + \frac{\Psi(N_t)}{\eta\sqrt{\tau}}\right) - \phi\left(d_1(\underline{\xi}^{VaR}) + \frac{\Psi(N_t)}{\eta\sqrt{\tau}}\right)\right)\right].$$

$$\tag{20}$$

Transforming the π's to the optimal portfolio weights on the stock x_t^S and the option x_t^O,

we have

$$x_t^S = \left(1 - \left(\frac{\Delta O}{\mu O_t} - \frac{O_S S_t}{O_t}\right)^{-1} \frac{O_S S_t}{O_t} \left(\frac{\sigma\left(1 - \frac{\lambda^Q}{\lambda}\right)}{\mu\eta} - 1\right)\right) \pi_t^{Z,VaR} \tag{21}$$

$$x_t^O = \left(\frac{\Delta O}{\mu O_t} - \frac{O_S S_t}{O_t}\right)^{-1} \left(\frac{\sigma\left(1 - \frac{\lambda^Q}{\lambda}\right)}{\mu\eta} - 1\right) \pi_t^{Z,VaR} \tag{22}$$

Consistent with Basak and Shapiro (2001), (16) reveals that the optimal time-t wealth consists of three components: a myopic component that maximizes Sharpe ratio and represents the optimal wealth of the B agent and two option components that correspond to the positions in the short put and the long put options, respectively. However, in contrast to Basak and Shapiro (2001), the option prices in the presence of jump risk do not immediately follow from the Black-Scholes option pricing formula, but rather are computed as the expectation of the Black-Scholes option prices conditional on jumps realized with respect to the jump risk factor.

(21) and (22) present the optimal portfolio weights on the stock x_t^S and the option x_t^O. Consistent with Liu and Pan (2003), the optimal weight on the option x_t^O is inversely proportional to its ability to disentangle the two exposure, which is characterized by the difference between the loading on diffusive risk and that on jump risk $\left(\frac{\Delta O}{\mu O_t} - \frac{O_S S_t}{O_t}\right)$; the larger the difference is, the more effective the option is in separating the two risk factors, the less it is needed, holding other things constant. An extreme case is that if the price of an option has equal sensitivities to both infinitesimal and large changes in the price of the underlying stock, that is, $g_S = \frac{\Delta g}{\mu S_t}$, then it is incapable of providing separate exposures to both the diffusive and jump risks and therefore becomes redundant.

The second term in (22) shows the demand for the option is ultimately determined by the risk-and-return tradeoff. Irrespective of whether the option is able to disentangle the two risk factors, the investor is unwilling to hold it, if the two risk factors are equally attractive, which implies,

$$\frac{\lambda^Q}{\lambda} = 1 - \frac{\mu\eta}{\sigma}. \tag{23}$$

(22) indicates that under such a constraint, the optimal weight on the option x_t^O is zero. The intuition is that, when the agent finds two risk factors equally attractive, her willingness to disentangle them diminishes, leading to a zero option holding as well. However, empirical evidence indicates that the coefficient $\frac{\lambda^Q}{\lambda}$ is much higher than $1 - \frac{\mu\eta}{\sigma}$ (see, for example Pan (2002)). This implies that the demand for the option is nonzero, because it helps the investor to have more jump risk premium.

In addition to looking at stock holding and option holding separately, it is also tempting to examine the stock-option ratio, which is given by,

$$\frac{x_t^S}{x_t^O} = \frac{\left(1 - \left(\frac{\Delta g}{\mu O_t} - \frac{g_S S_t}{O_t}\right)^{-1} \frac{g_S S_t}{O_t} \left(\frac{\sigma\left(1 - \frac{\lambda^Q}{\lambda}\right)}{\mu\eta} - 1\right)\right)}{\left(\frac{\Delta g}{\mu O_t} - \frac{g_S S_t}{O_t}\right)^{-1} \left(\frac{\sigma\left(1 - \frac{\lambda^Q}{\lambda}\right)}{\mu\eta} - 1\right)}. \tag{24}$$

The stock-option ratio reflects the composition of the portfolio and serves to capture the hedging role of options. For example, in the absence of jump risk premium, options can only be utilized as hedging vehicles . Thus, a lower stock-option ratio implies a stronger demand for options for the purpose of hedging. Note that the stock-option ratio does not depend on the risk averse coefficient γ.

[Insert Figure 3 about here.]

Figure 3 depicts the optimal wealth of the VaR agent, the B agent and the PI agent at time t. The optimal prehorizon wealth of the VaR agent behaves similarly to that of the B agent in both good and bad states. In contrast, in the intermediate region, the VaR agent's wealth does not coincide with the PI agent's wealth, because she just begins to insure against intermediate state. An interesting observation is that the presence of jump risk induces the VaR agent to maintain higher wealth in the intermediate and worst states at the expense of lower wealth in the favorable states. More importantly, her wealth goes above the B agent's wealth in more of bad states, thereby alleviating the problem with VaR identified by Basak and Shapiro (2001) that under the VaR constraints, risk managers optimally take larger exposure to risky assets in unfavorable states and incur larger losses than non-risk managers.

[Insert Figure 4 about here.]

Figure 4 illustrates the optimal time-t equity exposure of the VaR agent, the B agent and the PI agent relative to the B agent's equity exposure. In the two extreme states, the VaR agent acts like the B agent. In between, her equity exposure first moves similarly to the PI agent and decreases with ξ_t. Then, she takes increasingly large equity exposure, as the states worsen. Finally, when ξ_T is sufficiently large, the VaR agent's equity exposure again goes back toward the benchmark case. The fluctuation of the VaR agent's equity exposure is due to insuring against the intermediate states: when ξ_t is not so high, she chooses to take a large equity exposure to achieve portfolio insurance level $\underline{W}$. On the contrary, when ξ_t is already very high, all hope is gone and she simply behaves like the B agent. On the other hand, it is obvious that the jump component in the pricing kernel causes the VaR agent to deviate more from the benchmark case and her equity exposure to fluctuate in a larger region. The right panel shows the the optimal time-t equity exposure of the VaR agent for different α in the Basak and Shapiro case. Interestingly, comparison between the left panel and the right panel reveals that the effect of increasing jump risk premium is similar to that of decreasing α in the benchmark case; both make the deviation from the benchmark spread to a larger region of ξ_t. The intuition is that both tighter constraint (lower α) and riskier environment (higher λ^Q/λ) drive the VaR agent to invest more conservatively in that she has to fully insure against a larger intermediate region. Therefore, larger jump amplitude of ξ_t induces the VaR agent to adjust her asset allocation towards the PI case, making the properties of VaR more desirable from risk management point of view.

[Insert Table 3 about here.]

To quantitatively study the optimal portfolio strategies under the VaR constraint in a jump-diffusion model, we carry out some numerical experiments. For the purpose of comparison, we use the same setting as in Liu and Pan (2003). The risk-free rate is fixed at $r = 0.05$, and the diffusive risk premium and the market volatility are set equal to 0.08 and 0.15 respectively. Here we consider three different jump cases: $\mu = -10\%$ jumps once every 10 years, $\mu = -25\%$ jumps once every 50 years and $\mu = -50\%$ jumps once every 200 years. We investigate

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the cross-sectional variation of optimal stock and option strategies with respect to both jump-risk premium and option characteristics, including moneyness and time to expiration. As mentioned before, the out-of-the-money (OTM) put options are effective in distinguishing two risk factors. Put differently, the difference between the loading on diffusive risk and that on jump risk $\left(\frac{\Delta O}{\mu O_t} - \frac{O_S S_t}{O_t} \right)$ is larger for deep OTM put options. Thus we incorporate one at-the-money (ATM) option and two OTM options into our analysis to explore such effect.

Table 3 reports the optimal portfolio strategies under the VaR constraint with a one-month put option. If jump risk is not being compensated ($\frac{\lambda^Q}{\lambda} = 1$), the agent simply uses the stock to obtain the optimal exposure to diffusive risk. As the stock is suffering from negative jumps, the investor has to hold some put options to hedge such risk (see the second term in (21) for the delta hedging role played by the options). Therefore, in the absence of the jump risk premium, the option position only depends on its ability in disentangling two risk factors and no negative position would be taken since bearing jump risk is not rewarded. (see the first term in the second bracket in (22)). With the VaR constraint, however, one takes larger equity position as compared to Liu and Pan (2003) and correspondingly the option holding is increased to hedge the jump risk.

In the presence of positive jump risk premium, the agent switches to shorting options as the jump risk premium becomes sufficiently large, which is consistent with a variety of empirical evidence such as Driessen and Maenhout (2007). This is because jump risk becomes more attractive than diffusive risk. However, she keeps holding short position in option until jumps become much rarer and larger. The observation that when jumps become much rarer and larger, the agent is reluctant to get large exposure to jump risk, even though there is a high risk premium, is a direct consequence of the fact that the agent needs to prepare for those worst scenarios to satisfy the VaR constraint. Moreover, cross-sectional analysis of a variety of option features reveals that the option holding in absolute value decreases with moneyness, as deeper OTM put options are more effective in disentangling the exposure to two risk factors. To further explore the hedging demand for put options in the VaR framework, we also study the stock-option ratio, which is illustrated in Panel B. Comparison with Liu and Pan (2003)

reveals that the institutional investor puts more wealth in options when they are confronted with the VaR constraint, which is likely to be driven by higher hedging demand.

4 Conclusion

This paper studies optimal portfolio and wealth policies of an institutional investor with the VaR constraints in a jump-diffusion model. We reveal several interesting results. First, overlooking or underestimating the jump risk factor could lead to the failures of many financial institutions to satisfy the VaR constraint in the recent financial crisis. This finding is consistent with the criticism on normality assumptions that are widely used in applications of the VaR in practice raises the importance of incorporating jump risks into the VaR risk management framework. Second, the presence of the jump risk factor drives the institutional investor to behave like the portfolio insurance manager, alleviating the problem with VaR identified by Basak and Shapiro (2001) that the VaR risk manager incurs larger losses than non risk manager in worst scenarios. The conservative investment behavior induced by jump risks might be valuable and useful from a regulation point of view. Third, put options play an important role in hedging jump risk under the VaR constraint. In particular, the higher demand for put options generated by the VaR constraint may provide an explanation for the empirically observed popularity of index puts.

To the best of our knowledge, this paper is the first to consider the jump risk in a VaR risk management framework. Although the jump risk has been extensively studies in the literature, the analysis of its effect on the risk management is largely missing. This is exactly where this paper comes in. In addition, we also explore the hedging role of options and provide some insights into the option trading strategies from a risk management point of view. As the precision of parameter values has substantial influence on the success of the VaR risk management, taking into account model uncertainty in the VaR framework is an interesting extension to this paper.

A Proof of Proposition 1

As we completely follow Basak and Shapiro (2001) in deriving Proposition 1, one can see the Proof of Proposition 1 in the Appendix of Basak and Shapiro (2001) for reference.

B Proof of Proposition 2

With complete market assumption, Itô's lemma implies that $\xi_t W_t$ is a martingale:

$$
\begin{aligned}
W_t^{VaR} &= \mathrm{E}_t\left[\frac{\xi_T}{\xi_t} W_T^{VaR}\right] \\
&= \mathrm{E}_t\left[\frac{\xi_T}{\xi_t}(y\xi_t)^{-\frac{1}{\gamma}}|\xi_T < \underline{\xi}^{VaR}\right] + \mathrm{E}_t\left[\frac{\xi_T}{\xi_t}\underline{W}|\underline{\xi} < \xi_T < \bar{\xi}^{VaR}\right] \\
&\quad + \mathrm{E}_t\left[\frac{\xi_T}{\xi_t}(y\xi_T)^{-\frac{1}{\gamma}}|\xi_T > \bar{\xi}^{VaR}\right]
\end{aligned}
\tag{25}
$$

We compute each term in (25) separately,

$$
\mathrm{E}_t\left[\frac{\xi_T}{\xi_t}(y\xi_T)^{-\frac{1}{\gamma}}|\xi_T < \underline{\xi}^{VaR}\right] = \frac{1}{y^{\frac{1}{\gamma}}\xi_t}\mathrm{E}_t\left[\mathrm{E}_t\left[\xi_T^{1-\frac{1}{\gamma}}|\xi_T < \underline{\xi}^{VaR}, \sigma(N_T - N_t)\right]\right]
\tag{26}
$$

where the conditioning σ-algebra $\sigma(N_T - N_t)$ is the one generated by the random variable $(N_T - N_t)$. To avoid confusion with the volatility parameter σ, we simply write it as $(N_T - N_t)$ in what follows.

It is easy to see $\ln \xi_T$ follows normal distribution conditional on both $\mathcal{F}_t$ and $(N_T - N_t)$,

$$
\ln \xi_T|\mathcal{F}_t, (N_T - N_t) \sim \mathcal{N}(\ln \xi_t - (r + \frac{1}{2}\eta^2)\tau - \Psi(N_t), \eta^2\tau).
\tag{27}
$$

Let $A = \ln \xi_t - (r + \frac{1}{2}\eta^2)\tau - \Psi(N_t)$. Then, (27) implies

$$
\mathrm{E}_t \left[\xi_T^{1-\frac{1}{\gamma}} | \xi_T < \underline{\xi}^{VaR}, (N_T - N_t) \right]
$$

$$
= \int_{-\infty}^{\ln \underline{\xi}^{VaR}} \mathrm{e}^{\frac{\gamma-1}{\gamma} \ln \xi_T} \frac{1}{\sqrt{2\pi\tau}\eta} \mathrm{e}^{-\frac{(\ln \xi_T - A)^2}{2\eta^2\tau}} \, \mathrm{d} \ln \xi_T
$$

$$
= \exp\left(\left(\frac{\gamma-1}{\gamma}\right)^2 \frac{\eta^2}{2}\tau + \frac{\gamma-1}{\gamma} A \right) \int_{-\infty}^{\ln \underline{\xi}^{VaR}} \frac{1}{\sqrt{2\pi\tau}\eta} \mathrm{e}^{-\frac{\left(\ln \xi_T - \left(A + \eta^2\left(1-\frac{1}{\gamma}\right)\tau\right)\right)^2}{2\eta^2\tau}} \, \mathrm{d} \ln \xi_T
$$

$$
= \frac{\mathrm{e}^{\Gamma_t}}{\xi_t^{\frac{1}{\gamma}-1}} \mathrm{e}^{\frac{1-\gamma}{\gamma}\Psi(N_t)} \mathcal{N}\left(d_1(\underline{\xi}^{VaR}) + \frac{\Psi(N_t)}{\eta\sqrt{\tau}} \right) \tag{28}
$$

Substituting (28) into (25) yields,

$$
\mathrm{E}_t \left[\frac{\xi_T}{\xi_t}(y\xi_T)^{-\frac{1}{\gamma}} | \xi_T < \underline{\xi}^{VaR} \right] = \frac{1}{y^{\frac{1}{\gamma}}\xi_t} \mathrm{E}_t \left[\frac{\mathrm{e}^{\Gamma_t}}{\xi_t^{\frac{1}{\gamma}-1}} \mathrm{e}^{\frac{1-\gamma}{\gamma}\Psi(N_t)} \mathcal{N}\left(d_1(\underline{\xi}^{VaR}) + \frac{\Psi(N_t)}{\eta\sqrt{\tau}} \right) \right]
$$

$$
\approx \frac{\mathrm{e}^{\Gamma_t}}{(y\xi_t)^{\frac{1}{\gamma}}} \left\{ 1 - \mathrm{E}_t \left[\mathrm{e}^{\frac{1-\gamma}{\gamma}\Psi(N_t)} \mathcal{N}\left(-d_1(\underline{\xi}^{VaR}) - \frac{\Psi(N_t)}{\eta\sqrt{\tau}} \right) \right] \right\}, \tag{29}
$$

where the approximation follows from the second order Taylor approximation. One can easily calculate the remaining two terms in (25) in a similar fashion,

$$
\mathrm{E}_t \left[\frac{\xi_T}{\xi_t}\underline{W} | \underline{\xi}^{VaR} < \xi_T < \bar{\xi}^{VaR} \right] = \mathrm{e}^{-r\tau}\underline{W}\mathrm{E}_t \left[\mathrm{e}^{-\Psi(N_t)} \left(-\mathcal{N}\left(d_2(\underline{\xi}^{VaR}) + \frac{\Psi(N_t)}{\eta\sqrt{\tau}} \right) \right. \right.
$$

$$
\left. \left. + \mathcal{N}\left(d_2(\bar{\xi}^{VaR}) + \frac{\Psi(N_t)}{\eta\sqrt{\tau}} \right) \right) \right] \tag{30}
$$

$$
\mathrm{E}_t \left[\frac{\xi_T}{\xi_t}(y\xi_T)^{-\frac{1}{\gamma}} | \xi_T > \bar{\xi}^{VaR} \right] \approx \frac{\mathrm{e}^{\Gamma_t}}{(y\xi_t)^{\frac{1}{\gamma}}} \mathrm{E}_t \left[\mathrm{e}^{\frac{1-\gamma}{\gamma}\Psi(N_t)} \mathcal{N}\left(-d_1(\bar{\xi}^{VaR}) - \frac{\Psi(N_t)}{\eta\sqrt{\tau}} \right) \right] \tag{31}
$$

Summing up (29), (30), (31), we obtain (16).

Applying Itô's lemma to (16), we can easily get (17) and (18).

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Table 1: Classification of States for Optimal Horizon Wealth under the VaR Constraint

λ	λ^Q	$\underline{\xi}^{VaR}$	$\bar{\xi}^{VaR}$	Pr^{VaR}	Pr^{PI}	y^{VaR}	y^B	y^{PI}
BS(2001)		0.99	2.23	37.0%	40.5%	1.12	1	1.15
1	1	0.99	2.23	37.0%	40.5%	1.12	1	1.15
1	1.5	0.91	3.51	37.7%	43.6%	1.22	1	1.31
1	2	0.83	5.98	33.1%	42.2%	1.34	1	1.60
1	3	0.82	10.38	21.4%	26.5%	1.35	1	1.69
1.5	2	0.93	3.12	38.5%	43.7%	1.20	1	1.27
1.5	3	0.80	7.27	30.8%	40.1%	1.39	1	1.74
2	3	0.86	4.50	37.3%	44.8%	1.30	1	1.46

The table reports the classification of the state of the world for optimal horizon wealth under the VaR constraint. Pr^{VaR} is the probability that the VaR agent's terminal wealth is under portfolio insurance. Pr^{PI} is the probability that the PI agent's terminal wealth is under portfolio insurance. The parameter values are: $\xi_0 = 1$, $r = 0.05$, $\eta = 0.4$, $T = 1$.

Table 2: Certainty Equivalent Wealth under the VaR constraint

λ	λ^Q	$\gamma = 0.5$	$\gamma = 1$	$\gamma = 2$
Basak and Shapiro		1.161	1.120	1.091
1	1	1.161	1.120	1.091
1	1.5	1.271	1.196	1.139
1	2	1.572	1.404	1.268
1	3	3.265	2.462	1.913
1.5	2	1.242	1.174	1.125
1.5	3	1.888	1.597	1.381
2	3	1.402	1.280	1.189

The table reports the certainty equivalent wealth under the VaR constraint. The initial wealth is assumed to be one. Other parameter values are: $\xi_0 = 1$, $r = 0.05$, $\eta = 0.4$, $T = 1$.

Table 3: Optimal Strategies under the VaR constraint (One Month Put Option)

Jump Cases			Case I $\mu = -0.10$ Every 10 years			Case II $\mu = -0.25$ Every 50 years			Case III $\mu = -0.50$ Every 200yr		
		λ^Q/λ	ATM	5% OTM	10%OTM	ATM	5% OTM	10%OTM	ATM	5% OTM	10%OTM
					Panel A: Optimal Equity and Option Strategies						
$\gamma = 0.5$											
	1	x_t^S	45.328	25.852	21.609	39.085	23.132	20.370	37.714	22.699	20.260
		x_t^O	0.875	0.138	0.035	0.655	0.075	0.013	0.602	0.062	0.008
	1.5	x_t^S	9.657	17.319	18.988	28.180	21.277	20.083	32.652	21.910	20.167
		x_t^O	-0.343	-0.054	-0.014	0.283	0.032	0.006	0.430	0.044	0.006
	2	x_t^S	-23.682	9.239	16.409	17.462	19.407	19.743	27.614	21.111	20.056
		x_t^O	-1.468	-0.233	-0.059	-0.079	-0.009	-0.002	0.260	0.027	0.003
$\gamma = 3$											
	1	x_t^S	2.862	1.632	1.364	2.457	1.454	1.281	2.371	1.427	1.274
		x_t^O	0.055	0.009	0.002	0.041	0.005	0.001	0.038	0.004	0.001
	1.5	x_t^S	0.618	1.108	1.215	1.777	1.341	1.266	2.054	1.378	1.269
		x_t^O	-0.022	-0.003	-0.001	0.018	0.002	0.000	0.027	0.003	0.000
	2	x_t^S	-1.542	0.602	1.069	1.106	1.229	1.251	1.740	1.330	1.263
		x_t^O	-0.096	-0.015	-0.004	-0.005	-0.001	0.000	0.016	0.002	0.000
$\gamma = 5$											
	1	x_t^S	1.133	0.646	0.540	0.976	0.578	0.509	0.942	0.567	0.506
		x_t^O	0.022	0.003	0.001	0.016	0.002	0.000	0.015	0.002	0.000
	1.5	x_t^S	0.245	0.440	0.482	0.706	0.533	0.503	0.816	0.548	0.504
		x_t^O	-0.009	-0.001	0.000	0.007	0.001	0.000	0.011	0.001	0.000
	2	x_t^S	-0.612	0.239	0.424	0.440	0.489	0.497	0.691	0.529	0.502
		x_t^O	-0.038	-0.006	-0.002	-0.002	0.000	0.000	0.007	0.001	0.000
					Panel B: Optimal Equity to Option Ratio($\frac{x_t^S}{x_t^O}$)						
		λ^Q/λ									
		1	51.790	187.366	622.519	59.713	308.660	1526.342	62.629	368.626	2541.793
		1.5	-28.156	-319.516	-1385.107	99.723	657.468	3470.101	75.882	498.029	3535.261
		2	16.132	-39.727	-277.493	-219.833	-2132.918	-12081.202	106.136	793.796	5806.165

This table reports the optimal portfolio strategies under the VaR constraint in a jump-diffusion model. The options used are one-month European options with different moneyness. OTM denotes out-of-the-money options, and ATM denotes at-the-money options. Panel A shows the stock and option strategies and Panel B reports the stock-option ratio.
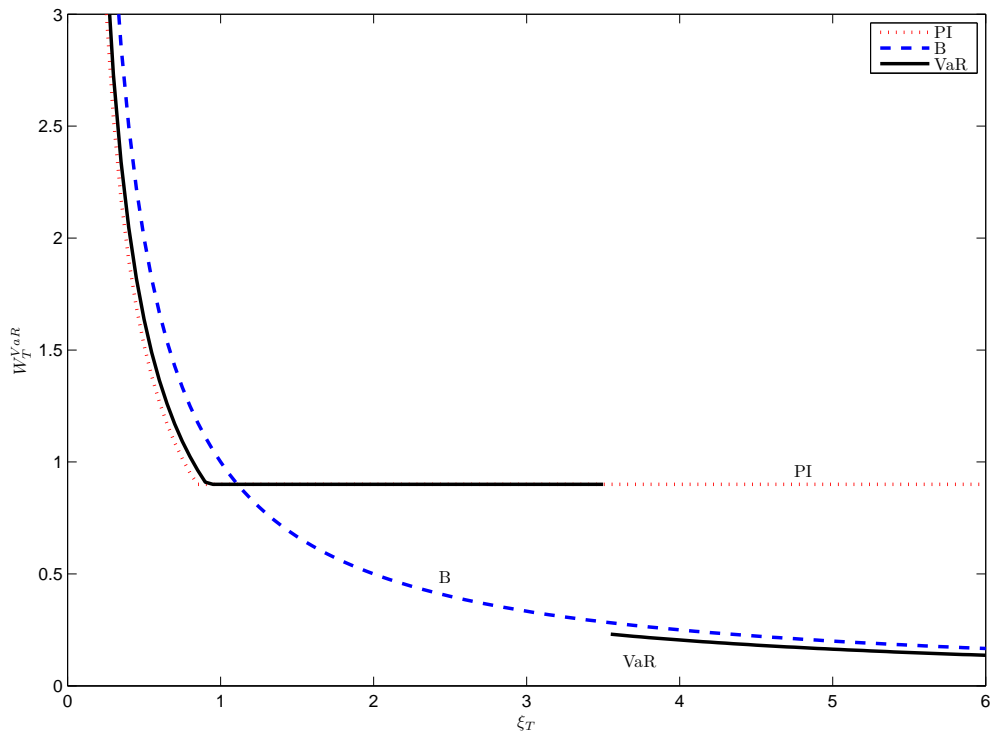
Figure 1: Optimal horizon wealth of three types of agents. The figure plots the optimal horizon wealth of the VaR agent (solid line), the benchmark agent (dashed line) and the portfolio insurance agent (dotted line). The parameter values are: $\lambda = 1$, $\lambda^Q = 1.5$, $\xi_0 = 1$, $r = 0.05$, $\eta = 0.4$, $\sigma_S = 0.18$, $T = 1$, $t = 0.5$, $\gamma = 1$, $\alpha = 0.01$, $W_0 = 1$, $\underline{W} = 0.9$.
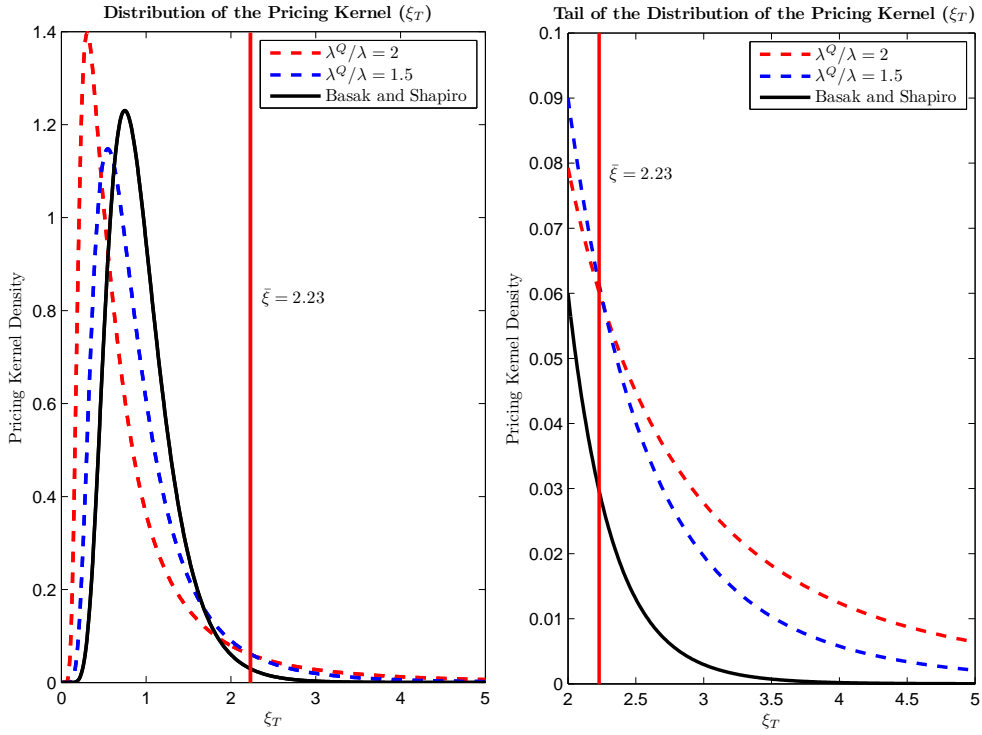
Figure 2: Distribution of the pricing kernel at horizon. The left plots the distribution of the pricing kernel at horizon for different jump parameters, while the right panel plots the upper tail of the distribution. In both panels, the black solid line is for $\lambda^Q/\lambda = 1$, the blue dashed line is for $\lambda^Q/\lambda = 1.5$ and the red dashed line is for $\lambda^Q/\lambda = 2$. The red solid line is for $\bar{\xi}$ in the Basak and Shapiro case. λ is fixed at 1, while λ^Q varies in different cases. Other parameter values are: $\xi_0 = 1$, $r = 0.05$, $\eta = 0.4$, $T = 1$.
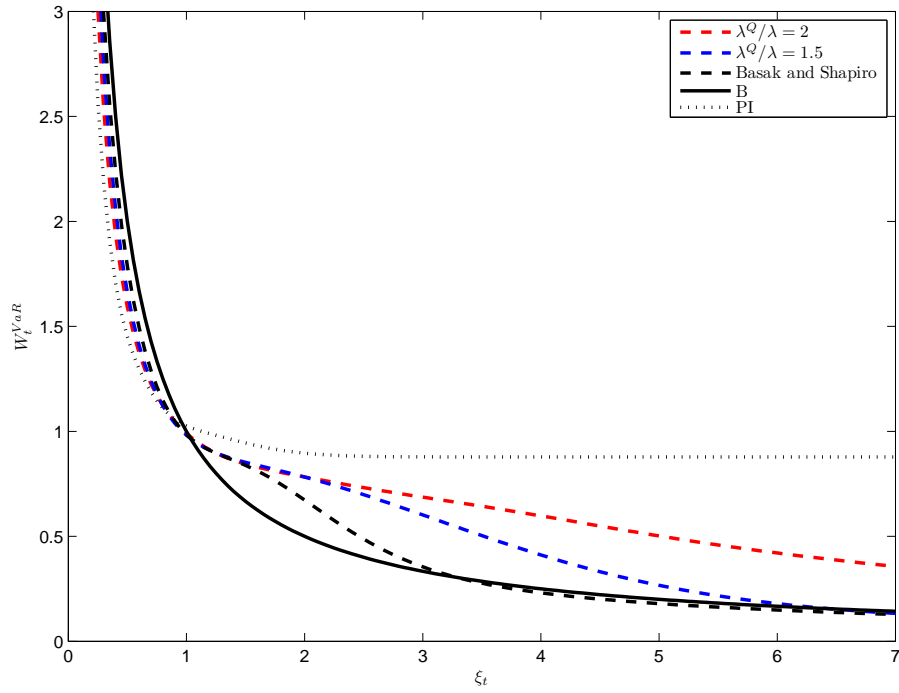
Figure 3: Optimal time-t wealth of different types of agents. The figure plots the optimal time-t wealth of the VaR agent, the benchmark agent and the portfolio insurance agent. The black solid line is for $\lambda^Q/\lambda = 1$, the blue dashed line is for $\lambda^Q/\lambda = 1.5$ and the red dashed line is for $\lambda^Q/\lambda = 2$. The solid line is for the benchmark case and the dotted line is for the portfolio insurance case. λ is fixed at 1, while λ^Q varies in different cases. Other parameter values are: $\xi_0 = 1$, $r = 0.05$, $\eta = 0.4$, $\sigma_S = 18$, $T = 1$, $t = 0.5$, $\gamma = 1$, $\alpha = 0.01$, $W_0 = 1$, $\underline{W} = 0.9$.
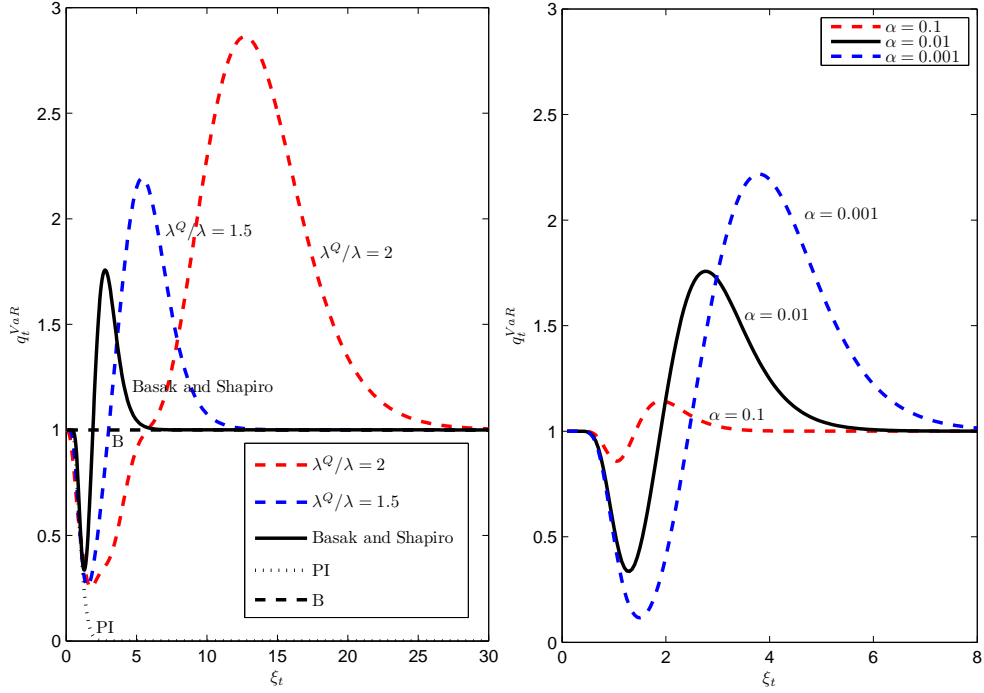
Figure 4: Optimal time-t equity exposure of different types of agents relative to the benchmark case. The left panel plots the optimal time-t equity exposure of the VaR agent, the benchmark agent and the portfolio insurance agent relative to the benchmark case. The right panel plots the optimal pre-horizon risk exposure of the VaR agent in the Basak and Shapiro case for different α for comparison purpose. In the left panel, the solid line is for $\lambda^Q/\lambda = 1$, the blue dashed line is for $\lambda^Q/\lambda = 1.5$ and the red dashed line is for $\lambda^Q/\lambda = 2$. λ is fixed at 1, while λ^Q varies in different cases. The black dashed line is for the benchmark case and the dotted line is for the portfolio insurance case. In the right panel, the solid line is for $\alpha = 0.01$, the blue dashed line is for $\alpha = 0.001$ and the red dashed line is for $\alpha = 0.1$. Other parameter values are: $\xi_0 = 1$, $r = 0.05$, $\eta = 0.4$, $\sigma_S = 18$, $T = 1$, $t = 0.5$, $\gamma = 1$, $\alpha = 0.01$, $W_0 = 1$, $\underline{W} = 0.9$.