Optimal Annuitization Decisions with Liquidity Constraints

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Abstract

We investigate the optimal annuitization strategy for retirees in the presence of liquidity constraints. A single annuitization opportunity at retirement is assumed. For given annuity income streams, we first solve the portfolio and consumption optimization problem under liquidity constraints in closed-form. Based on the analytical solution, we find the optimal annuitization levels for nominal, real, and variable annuities. Full annuitization is in general not optimal because of the non-marketability of future annuity income. The utility losses associated with liquidity constraints are highest for variable annuity in the case of high risk aversion and for real annuity in the case low risk aversion, while they are lowest for nominal annuity in most cases.

Keywords: Portfolio choice; Liquidity constraint; Household finance

JEL Codes: G11; D11

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1 Introduction

Economic theory has long established that life annuities, which regularly pay out until the policyholder dies, are valuable in providing protection against longevity risk and smoothing consumption. In a seminal contribution, Yaari (1965) shows that under some restrictive assumptions, such as complete annuity market and no bequest motive, people should fully annuitize their wealth because of the survival credit offered by life annuities. However, empirical evidence suggests that retirees seldom voluntarily annuitize when they approach retirement, which is known as annuity puzzle. One possible explanation is the irreversibility of annuity products: it deters annuitants from borrowing against future annuity income to smooth consumption. Liquidity problem arises if retirees annuitize a large fraction of their wealth at retirement. Moreover, the utility losses associated with liquidity constraints differ across different types of annuities, namely nominal, real and variable annuities, because they provide different exposures to inflation and equity risk and generate different income streams. In this paper, we investigate the optimal annuity purchase strategy for liquidity-constrained retirees. we quantify the utility costs of imperfect annuity market and compare the attractiveness of a variety of annuity products.

Liquidity constraints resulting from nonmarketable future income are of great interest and importance in the literature studying the life-cycle consumption and portfolio choice problem, because they impose substantial impacts on household behavior: in the presence of liquidity constraints, individuals have to sacrifice early consumption for a buffer stock against future liquidity shocks. Particularly in a stochastic world, liquidity constraints restrict not only the transfer of wealth across time, but also the transfer of wealth across states, thereby resulting in sizable utility losses for retirees. Nonetheless, solving the life-cycle optimization problems with liquidity constraints is far from easy, because the associated market incompleteness considerably undermines the analytical tractability. Theoretical advances in finding optimal portfolio and consumption strategy with labor income and liquidity constraints include He and Pages (1993),El Karoui and Jeanblanc-Picqué (1998), Koo (2002), Detemple and Serrat (2003), Dybvig and Liu (2010). These papers succeed in deriving closed-form solutions and thus facilitate the understanding of optimal consumption and investment strategies with nonmarketable income streams. To the best of our knowledge, none of the extant studies has applied this theory in a post-retirement context. However, liquidity constraints faced by retirees are not negligible and are particularly important for their annuitization decisions, because annuity products are irreversible or can be converted to liquid assets only at highly unfavorable prices. This paper extends the literature on portfolio choice under liquidity constraints by incorporating time varying mortality rates and inflation risk, and focuses on liquidity constraints in retirement.

Although annuity income is akin to labor income in terms of non-marketability, it has two distinct features. First, annuity income is not subject to unspanned idiosyncratic income risk, and consequently can cause market incompleteness only through its illiquidity. Second, unlike young agents in the working phase, those approaching retirement are largely capable of determining future income streams by choosing annuitization levels[1]. While mortality premium increases with annuitization levels, liquidity of wealth deteriorates simultaneously. The trade-off between the mortality credit and the welfare costs of liquidity constraints raises the importance of optimizing annuity purchase decisions for retirees. To cope with future liquidity constraints, retirees may deviate from full annuitization strategy in order to maintain a certain amount of financial wealth as a buffer stock, which might partially resolves annuity puzzle. Moreover, it is important to take into account the characteristics of annuity products in making annuitization decisions, because different types of annuities generate different income streams and therefore results in different strength of liquidity constraints. For example, the real income process associated with nominal annuity is decreasing, while that with real annuity keeps flat. Therefore, nominal annuity is less likely to be affected by liquidity constraints than real annuity.

Our main results are as follows. First, the welfare losses associated with liquidity constraints are economically sizable and individuals can be better off by deviating from full annuitization policy, which partially resolves the well-known "annuity puzzle". Second, the optimal annuitization level depends on the feature of the associated income process, the preference of retirees and the annuitization age. In general, liquidity constraints are more likely to bind when income stream is low in the early time periods and high in the late life, since the desire to borrow against future income for consumption smoothing is stronger. For this reason, nominal annuity, which produces decreasing real annuity income stream, is least affected by liquidity constraints. Although variable annuity features low initial payoff and high income growth, its attractiveness increases

[1]In some countries, annuitizing a fraction of retirement wealth is mandatory. Therefore, the choice of annuitization levels is not completely at the discretion of retirees.

as risk aversion goes down and even outweighs that of real annuity for households that are slightly risk-averse. The intuition behind is that the effects of liquidity constraints weaken when the annuity payoff stream matches better the optimal wealth dynamics in the absence of liquidity constraints. Specifically, households who are less risk-averse tend to invest more aggressively in stocks and enjoy higher consumption growth due to equity premium, which is line with the income process generated by variable annuity. In contrast, if the households purchase real annuity, they have to sacrifice early consumption to hold more risky assets and as a consequence incur substantial utility losses. Hence, individuals with lower risk-aversion prefer variable annuity to real annuity. In addition, the optimal annuitization level is increasing in the annuitization age, because mortality credit rises and liquidity constraints weaken as age grows and planning horizon decreases.

A large strand of literature has studied the optimal annuity choice. Brown, Mitchell, and Poterba (2001) show that both real annuities and variable annuities can be welfare-improving. They also point out that the attractiveness of variable annuities in their model stems from the exclusion of stocks in the asset menu and it is important to study the case where individuals have the ability to invest in more financial assets. Horneff, Maurer, Mitchell, and Dus (2008) find that integrating asset allocation and annuitization allows the retiree to benefit from both the equity premium and the mortality credit, which is attributed to the attractiveness of equity-linked phased withdrawal plans. Koijen, Nijman, and Werker (2011) study the optimal combination of different annuity products with stochastic interest risk and time varying risk premia. Our paper is different from the above studies in several aspects. First, while most of the previous papers restrict the asset menu of the retirees to cash, we allow the individuals to have access to other financial assets such as stocks and real bonds after annuitization. This is because ignoring active financial market participation overestimates the welfare gains from annuities and conceal the losses from liquidity constraints. Second, instead of focusing on expected payoffs, we highlight the liquidity constraints associated with different annuity products, which provide insights into portfolio choices from a different but important perspective. Third, most of the previous papers are built on complex numerical algorithms. By using framework from labor income literature, we obtain a closed-form solution to optimal consumption strategy for given annuity income streams based on which, deriving the optimal annuitization levels reduces to a one-dimensional search.

This paper contributes to the literature on several fronts. First, to the best of our knowledge, this paper is the first attempt to provide an explicit solution to the optimal consumption strategy with illiquid annuity income, which reduces computational burden and facilitates the understanding of the effects of liquidity constraints. Second, by considering different annuity products and a richer asset menu, we provide further insights into the retirement portfolio choice problem. In particular, we show that annuities providing low income in early periods but high income growth, such as variable annuities, are less attractive than their competitors for both moderately and highly risk-aversion retirees. This finding stands in stark contrast to several previous papers establishing that variable annuities are most appealing. (See, for example, Koijen, Nijman, and Werker (2011)). This distinction follows from the fact that in most of the previous literature, retirees are restricted to annuities and as a result can benefit from equity premium only through variable annuity, while in this study they can freely invest in stocks. Our paper sheds light on the behavior of retirees with active stock market participation, which is increasingly important as the retiree is more responsible for his retirement wealth in DC planes.

To maintain analytical tractability and focus on the impacts of liquidity constraints, we make simplifications. First, we allow the individual to purchase annuity only at retirement. Several papers argue that it might better to postpone the annuitization to reap equity premium first or to gradually annuitize wealth over the whole life-cycle. (See, for example, Milevsky and Young (2007) and Horneff, Maurer, and Stamos (2008).) Second, we impose no constraints on the individual's ability to short-sell risky assets or borrow cash, which means that the individual can adjust her positions in some assets by short-selling other marketable assets. This setting is controversial, but is common in most complex portfolio choice models with closed-form solutions. Besides, it emphasizes the better marketability of financial assets than that of annuities, which is one of the focuses of this paper. Third, we abstract from stochastic interest rate and time varying risk premia. Koijen, Nijman, and Werker (2011) show that conditioning the annuity choice on the state of the economy can be welfare-enhancing. Although we do not model these features, we can make some conjectures concerning their effects. If the positions of liquid assets can be freely tailored to the time varying state of the economy, financial assets are likely to be more attractive than irreversible annuity products. Thus, the importance of understanding the liquidity constraints associated with annuities, as emphasized in this paper, remains. Fourth, health shocks and bequest motives are absent. In principal, the former increases the liquidity demand and the latter decreases the mortality premium, both of which might decrease the optimal annuity holding and potentially explain the annuity puzzle (See, for example, Brown, Mitchell, and Poterba (2001), Turra and Mitchell (2004), Davidoff, Brown, and Diamond (2005), Inkmann, Lopes, and Michaelides (2010), Peijnenburg, Nijman, and Werker (2011) and Lockwood (2012)). We leave these extensions for future research.

The rest of this paper is organized as follows. Section 2 describes both financial market and annuity market under consideration and formulates the portfolio and consumption optimization problem. Section 3 introduces the stopping time approach and apply it to the determination of the optimal consumption strategy under liquidity constraints. Section 4 compares the effects of liquidity constraints on different annuity products. Section 5 concludes the paper and Appendix presents proofs.

2 The model

2.1 Financial market

The financial market model we consider is closely related to the model of Brennan and Xia (2002), who characterize the term structure with two state variables, the real interest rate and expected inflation. In contrast, to ensure analytical tractability, we abstract from time-variation of both state variables and leave the inflation risk entirely driven by unexpected shocks . As a consequence, there are only two risk factors in the economy, unexpected inflation and equity risk.

We assume that the commodity price level, Π, is given by,

$$\begin{aligned}\frac{\mathrm{d}\Pi}{\Pi} &= \pi \mathrm{d}t + \xi_S' \mathrm{d}z_S + \xi_U \mathrm{d}z_U, \\ &= \pi \mathrm{d}t + \xi' \mathrm{d}z \end{aligned} \tag{1}$$

where π is the constant instantaneous rate of inflation, z_S and z_U are two Brownian motions representing equity risk and unexpected inflation risk, respectively. The initial price level Π_0 is normalized to one. It is important to note that throughout this paper we denote nominal and real variables by uppercase and lowercase letters, respectively.

The investment opportunities depend on the real pricing kernel of the economy, m, which follows a diffusion process:

$$\begin{aligned}\frac{\mathrm{d}m}{m} &= -r\mathrm{d}t + \phi_S \mathrm{d}z_S + \phi_U \mathrm{d}z_U \\ &= -r\mathrm{d}t + \phi' \mathrm{d}z, \end{aligned} \tag{2}$$

where r is the constant interest rate and ϕ_i $(i = S, U)$ represents the constant loadings on the stochastic innovations in the economy and determines the market prices of risk, λ_S and λ_U, which are associated with innovations $\mathrm{d}z_S$ and $\mathrm{d}z_U$ respectively. Brennan and Xia (2002) shows that the nominal short-term risk-free rate R and the vector of market prices of risk $\lambda = [\lambda_S, \lambda_U]'$ are:

$$\begin{aligned}\lambda &= \rho(\xi - \phi), \\ R &= r + \pi - \xi'\lambda, \end{aligned} \tag{3}$$

where ρ is the correlation matrix of $\mathrm{d}z$.

To complete market, the investor's asset menu consists of three assets: a stock, an inflation-linked bonds, and a nominal cash account. The dynamics of nominal stock price and nominal return of the inflation-indexed bond are given by,

$$\frac{\mathrm{d}S}{S} = (R + \sigma_S \lambda_S)\mathrm{d}t + \sigma_S \mathrm{d}z_S \tag{4}$$

$$\frac{\mathrm{d}P}{P} = (r + \pi)\mathrm{d}t + \xi' \mathrm{d}z. \tag{5}$$

2.2 Annuity market

We consider three types of immediate annuity products, namely nominal annuities, inflation-linked annuities and variable annuities. Nominal annuities provide a constant nominal payoff stream for the rest of the annuitant's life. In contrast, the payments offered by inflation-linked annuities are constant in real terms and therefore can help the annuitant hedge against inflation risk. The third type is a so-called variable annuity, whose payments are linked to a broad equity index. Consequently, buying variable annuities allows the retiree to (possibly) benefit from equity risk premium, but in the meantime makes her retirement income more volatile. In the rest of this subsection, we follow Koijen, Nijman, and Werker (2010) in pricing the three types of annuity products.

We assume throughout that the prices of the annuities under consideration are actuarially fair. In addition, the longevity risk is assumed to be idiosyncratic and can be characterized by the mortality intensity of a representative retiree, which is given by,

$$\lambda_t = \lambda_0 \mathrm{e}^{\lambda_1 t} \tag{6}$$

The modeling of mortality rate is consistent with standard models in the literature, for example the Gompertz law of mortality. Let A^N and A^R denote the pricing factors for the nominal and inflation-linked annuities, respectively: a purchase of nominal annuity with a premium A^N and that of inflation-linked annuity with a premium A^R yield real rates of income, I_t^N and I_t^R, respectively, which are given by,

$$I_t^N = 1/\Pi_t, \tag{7}$$
$$I_t^R = 1. \tag{8}$$

Therefore, A^N and A^R are given by,

$$A^N = \mathrm{E}\left[\int_0^T \mathrm{e}^{-\int_0^t \lambda_s \mathrm{d}s} \frac{1}{\Pi_t} \frac{m_t}{m_0} \mathrm{d}t\right] = \int_0^T \mathrm{e}^{-Rt - \frac{\lambda_0}{\lambda_1}\left(\mathrm{e}^{\lambda_1 t} - 1\right)} \mathrm{d}t \tag{9}$$
$$A^R = \mathrm{E}\left[\int_0^T \mathrm{e}^{-\int_0^t \lambda_s \mathrm{d}s} \frac{m_t}{m_0} \mathrm{d}t\right] = \int_0^T \mathrm{e}^{-rt - \frac{\lambda_0}{\lambda_1}\left(\mathrm{e}^{\lambda_1 t} - 1\right)} \mathrm{d}t \tag{10}$$

Variable Annuity

Real payout at time t: $\mathrm{e}^{-ht} \frac{S_t}{\Pi_t S_0}$

Annuity price:

$$V = \int_0^T \mathrm{e}^{-ht - \frac{\lambda_0}{\lambda_1}\left(\mathrm{e}^{\lambda_1 t} - 1\right)} \mathrm{d}t \tag{11}$$

Buying variable annuity receives real income at time t: $y_t = \frac{W_0 S_t}{V \Pi_t S_0}$, where y_t is given by,

$$\begin{aligned} \mathrm{d}y_t &= y_t \left[(r-h)\mathrm{d}t - \hat{\xi}'\mathrm{d}z^Q\right] \\ &= y_t \left[(r-h+\phi'\rho\hat{\xi})\mathrm{d}t - \hat{\xi}'\mathrm{d}z\right] \end{aligned} \tag{12}$$

where $\hat{\xi} = (\xi_S - \sigma_S, \xi_U)'$.

F is given by,

$$F_t = -\frac{\mathrm{e}^{-h(T-t)} - 1}{h} \tag{13}$$

2.3 Retiree's preferences, income and constraints

We consider a representative agent, is endowed with an initial wealth $\hat{W}_0 > 0$ and has a maximum lifespan of T. She is allowed to annuitize her wealth only at the retirement, namely at time 0 in the model. Specifically, she converts a fraction of her initial wealth into a certain type of annuity product at time 0, which is denoted by π_A. The annuity type is given exogenously. Thus, her initial wealth is split into two parts: one is the annuity premium $\pi_A\hat{W}_0$, which yields an initial real annuity payout of $y_0 = \pi_A\hat{W}_0/(A)$ and the other one is the initial liquid wealth $W_0 = (1-\pi_A)\hat{W}_0$.

The retiree's preferences are represented by the CRRA utility function with relative risk aversion parameter γ. Her objective is to maximize,

$$\mathrm{E}\left[\int_0^T \mathrm{e}^{-\int_0^t(\lambda_u+\delta)\mathrm{d}u}\frac{\left(\frac{C_t}{\Pi_t}\right)^{1-\gamma}}{1-\gamma}\mathrm{d}t\right], \tag{14}$$

where δ is the subjective time discount factor.

Depending on the annuity type and annuitization level that the individual chooses at retirement, she receives a nonnegative real income process y_t for $t \in [0, T]$, which can be generalized as,

$$\begin{aligned} \frac{\mathrm{d}y_t}{y_t} &= (\mu_y - \phi'\rho\sigma_y)\mathrm{d}dt + \sigma_y'\mathrm{d}z, \\ &= \mu_y \mathrm{d}dt + \sigma_y'\mathrm{d}z^Q \end{aligned} \tag{15}$$

where $\sigma_y = (\sigma_{yS}, \sigma_{yU})'$ and $\mathrm{d}z^Q = [\mathrm{d}z_S^Q \mathrm{d}z_U^Q]$ is the vector of the stochastic innovations in the economy under the risk neutral measure Q.

In a complete market setting, the budget constraints for the liquidity-constrained retiree can be formulated as,

$$\mathrm{E}\left[\int_0^s \frac{m_t}{m_0}\frac{C_t}{\Pi_t}\mathrm{d}t\right] \leq \mathrm{E}\left[\int_0^s \frac{m_t}{m_0} y_t \mathrm{d}t\right] + \frac{W_0}{\Pi_0} \qquad \forall s \in [0, T], \tag{16}$$

and the budget constraint for the unconstrained retiree as,

$$\mathrm{E}\left[\int_0^T \frac{m_t}{m_0}\frac{C_t}{\Pi_t}\mathrm{d}t\right] \leq \mathrm{E}\left[\int_0^T \frac{m_t}{m_0} y_t \mathrm{d}t\right] + \frac{W_0}{\Pi_0}. \tag{17}$$

A comparison between (16) and (17) reveals that the former nests the latter, thereby leading to much restrictive constraints on the portfolio and consumption choice of the liquidity-constrained retiree. The intuition is that the liquidity-constrained individual must ensure that her liquid wealth remains nonnegative at all times over her life-cycle. In contrast, the unconstrained individual can borrow her future income to sustain current spending as long as her total wealth is nonnegative. Here total wealth is comprised of liquid wealth and the present value of her future income streams. Therefore, she has much more flexibility to smooth consumption than her counterpart.

3 Optimal consumption strategy with liquidity constraints

In this section we study the constrained consumption problem in the previous section. He and Pages (1993) and El Karoui and Jeanblanc-Picqué (1998) solved similar model for a case with infinite horizon. Detemple and Serrat (2003) solved a model with finite horizon. Our model is an extension of the model in Detemple and Serrat (2003). We take into account positive initial endowment, subjective time preference, time-varying mortality rate, and inflation risk, which might be more revelent in a life cycle setting. For detailed on the stopping time approach, please refer to He and Pages (1993), El Karoui and Jeanblanc-Picqué (1998), and Detemple and Serrat (2003). In the following we first talk about the general framework to solve these kind of constrained problems, then we give the solutions in our setting.

3.1 Duality and stopping time approach

To solve the optimization problem with liquidity constraint (16), the general framework is to first build a Lagrange function and then solve the dual of the prime problem, as in He and Pages (1993). It is shown in the literature that the portfolio choice problem with liquidity constraint is equivalent to a stopping time problem in which wealth is allocated in an optimal unconstrained manner over an endogenous random time period.

Following section 2.1 in Detemple and Serrat (2003), optimal consumption and consumption strategies can be obtained once the process of optimal constraint wealth is determined. First define $z = \eta m_t$, where η represent the Lagrange multiplier for the static budget constraint.

The optimal constraint wealth w_t can be written as a function of z

$$w_t(z) = w_t^u(z) + \Omega_t(z) \tag{18}$$

where w_t^u is the optimal unconstrained wealth and Ω_t is an American put option written on the negative part of unconstrained wealth

$$\Omega_t(z) = \sup_{\tau \in [t,T]} \mathrm{E}_t \left[\max \left\{ -\frac{m_\tau}{m_t} w_t^u(z), 0 \right\} \right] \tag{19}$$

With these notations, we present the solution of optimal consumption strategies with and without liquidity constraints in the following. A more detailed description of this method is in section 3.2.3.

3.2 Solution of the life-cycle portfolio and consumption choice problem

3.2.1 Unconstrained problem

We first solve the unconstrained consumption and portfolio optimization problem with time-varying mortality rate under inflation.

$$\max_{C_t} \quad \mathrm{E}\left[\int_0^T \mathrm{e}^{-\int_0^t \hat{\delta}_u \mathrm{d}u} \frac{\left(\frac{C_t}{\Pi_t}\right)^{1-\gamma}}{1-\gamma} \mathrm{d}t\right] \tag{20}$$

$$\text{s.t.} \quad \mathrm{E}\left[\int_0^T \frac{m_t}{m_0} \frac{C_t}{\Pi_t} \mathrm{d}t\right] \leq \mathrm{E}\left[\int_0^T \frac{m_t}{m_0} y_t \mathrm{d}t\right] + \frac{W_0}{\Pi_0} \tag{21}$$

where $\hat{\delta}_u = \delta + \lambda_u$, W_0 is the initial financial. It is easy to see that the optimal consumption strategy is given by,

$$c_t = \left(\psi \mathrm{e}^{\int_0^t \hat{\delta}_u \mathrm{d}u} m_t\right)^{-\frac{1}{\gamma}} \tag{22}$$

where ψ is the Lagrange multiplier. Using budget constraint (21), we can solve for ψ,

$$\psi = \left(\frac{\frac{W_0}{\Pi_0} + y_0 F_0}{G_0}\right)^{-\gamma} \tag{23}$$

where G and F are deterministic function of time and are given by,

$$G_t = \int_t^T \exp\left\{\int_t^s -\frac{\hat{\delta}_u}{\gamma} \mathrm{d}u - \left(1 - \frac{1}{\gamma}\right) r(s-t) + \frac{1}{2}\left(\frac{1}{\gamma} - 1\right) \frac{1}{\gamma} \phi' \rho \phi (s-t)\right\} \mathrm{d}s \tag{24}$$

$$F_t = \frac{\mathrm{e}^{(\mu_y - r)(T-t)} - 1}{\mu_y - r} \tag{25}$$

The indirect utility at time 0 is given by:

$$J_0^u = \frac{\left(\frac{W_0}{\Pi_0} + y_0 F_0\right)^{1-\gamma} G_0^\gamma}{1-\gamma} \tag{26}$$

3.2.2 Constrained problem

First, notice that the optimal unconstrained real wealth process (w_t^u) is

$$w_t^u = c_t^u G_t - y_t F_t, \tag{27}$$

where c_t^u is unconstrained real consumption at time t.

We now define the unconstrained consumption-to-income ratio process ($c_t^* = c_t^u / y_t$), which can be written as

$$\frac{\mathrm{d}c_t^*}{c_t^*} = \mu^* \mathrm{d}t + \sigma_c \mathrm{d}\widetilde{z}_t^y, \tag{28}$$

where $\widetilde{z}_t^y$ is a one-dimensional standard Brownian motion. The time-varying drift term μ^* is given by,

$$\mu^* = \frac{1}{\gamma}\left[-\hat{\delta}_t + r + \frac{1}{2}\left(\frac{1}{\gamma} - 1\right)\phi' \rho \phi\right] - \mu_y. \tag{29}$$

and the volatility term σ_c is

$$\sigma_c = \sqrt{\left(\sigma_y + \frac{\phi}{\gamma}\right)' (\rho. \times \rho) \left(\sigma_y + \frac{\phi}{\gamma}\right)} \tag{30}$$

where $\rho. \times \rho$ is the entry-by-entry multiplication.

The optimal constrained wealth consists of two parts. The first part is the unconstrained liquidity wealth (w_t^u) and the second is an American put option written on w_t^u with null strike. From representation for w_t^u (equation (27)) and the representation for c_t^* (equation (28)), we can see that

$$\max\{w_t^u, 0\} = y_t G_t \max\{c_t^*, \frac{F_t}{G_t}\} \tag{31}$$

which implies that conditioning on income at t, the value of buffer stock is determined by c_t^*. Using the dynamics of c_t^*, the following theorem gives the optimal exercise boundary B_t and the constrained consumption strategy.

Theorem 1. *For all $t \in [0,T]$, optimal constrained wealth is,*

$$w_t = y_t V(c_t^*, t) + y_t \Omega\left(c_t^*; B(\cdot)\right) \tag{32}$$

where $y_t V(c_t^*, t)$ represents the unconstrained wealth and $y_t \Omega\left(c_t^*; B(\cdot)\right)$ represents the buffer stock for liquidity constraints (the early exercise premium (EEP) of the American put option on the unconstrained wealth). We have

$$V(c_t^*, t) = c_t^* G_t - F_t \tag{33}$$

$$\begin{aligned}\Omega\left(c_t^*; B(\cdot)\right) &= \Omega\left(c_t^*, 1; B(\cdot)\right)\\ &= \left[\int_t^T \mathrm{e}^{(\mu_y - r)(v-t)} \Phi\left(c_t^*, B_v, t, v\right) \mathrm{d}v\right]\end{aligned} \tag{34}$$

with

$$\begin{aligned}\Phi\left(c_t^*, B_v, t, v\right) =& N\left(-d(c_t^*, B_v, t, v) + \sigma_c \sqrt{v-t}\right)\\ &- z_t^y \mathrm{e}^{\int_t^v \mu_s^* \mathrm{d}s} N\left(-d(c_t^*, B_v, t, v)\right)\end{aligned} \tag{35}$$

and

$$d(c_t^*, B_v, t, v) = \frac{\log(c_t^*/B_v) + \int_t^v \left(\mu_s^* + \frac{1}{2}\sigma_c^2\right) \mathrm{d}s}{\sigma_c \sqrt{v-t}} \tag{36}$$

The optimal exercise boundary B is deterministic and solves the recursive integral equation

$$\Omega\left(B_t; B(\cdot)\right) = F_t - B_t G_t \tag{37}$$

with terminal condition $B_T = 1$.

Optimal real consumption-to-income ratio c_t^* is fully determined by the boundary B_t and the real pricing kernel m_t:

$$c_t^* = \left((\psi^* - \varsigma_t^*) \mathrm{e}^{\int_0^t \hat{\delta}_u \mathrm{d}u} m_t\right)^{-\frac{1}{\gamma}} \tag{38}$$

with

$$\varsigma_t^* = \left(\psi^* - \inf_{v\in[0,t]} \frac{(B_v y_v)^{-\gamma}}{e^{\int_0^v \hat{\delta}_u \mathrm{d}u} m_v}\right)^+ \tag{39}$$

where ψ^* is the optimal initial multiplier which solves the equation

$$V\left(\frac{(\psi^* m_0)^{-\frac{1}{\gamma}}}{y_0}, t\right) + \Omega\left(\frac{(\psi^* m_0)^{-\frac{1}{\gamma}}}{y_0}; B(\cdot)\right) = \frac{w_0}{y_0} \tag{40}$$

To understand the optimal constrained consumption strategy in equation (38), we can compare it to its unconstrained counterpart in equation (22). Without liquidity constraints, consumption is determined by a constant multiplier ψ, which is related to the shadow price for the individual's optimal consumption plan. With liquidity constraints, however, the shadow prices process is not constant. The individual consumes according to the initial multiplier ψ^*, until at a random time t the boundary is reached ($\varsigma_t^* > 0$). Then the multiplier changes to the value implied by the boundary ($\inf_{v\in[0,t]} \frac{(B_v y_v)^{-\gamma}}{e^{\int_0^v \hat{\delta}_u \mathrm{d}u} m_v}$). The individual consume according to the new multiplier until the boundary is reached at a second time. This pattern repeats until the terminal date.

3.2.3 Properties of boundaries

In this section we discuss some properties of the optimal exercise boundaries, which facilitate the understanding the optimal consumption strategies. In the martingale approach, consumptions rates are determined by the Lagrange multiplier and pricing kernel. The constrained initial consumption cannot be larger than the unconstraint initial consumption, since a portion of liquid wealth is put aside as buffer stock to meet future liquidity constraints. In our framework, the value of the buffer stock equals the price of an American put option written on the negative part of the unconstrained wealth process, and the exercise of the put amounts to resetting the multiplier. The boundary determines the time at which immediate exercise of the put option is optimal. The initial consumption-to-income ratio is either above the boundary (positive initial endowment) or on the boundary (zero initial endowment). The investor continues to consume according to the initial multiplier as if there is no constraint until the unconstrained wealth becomes sufficiently negative that the sum of unconstrained wealth and buffer stock reduces to

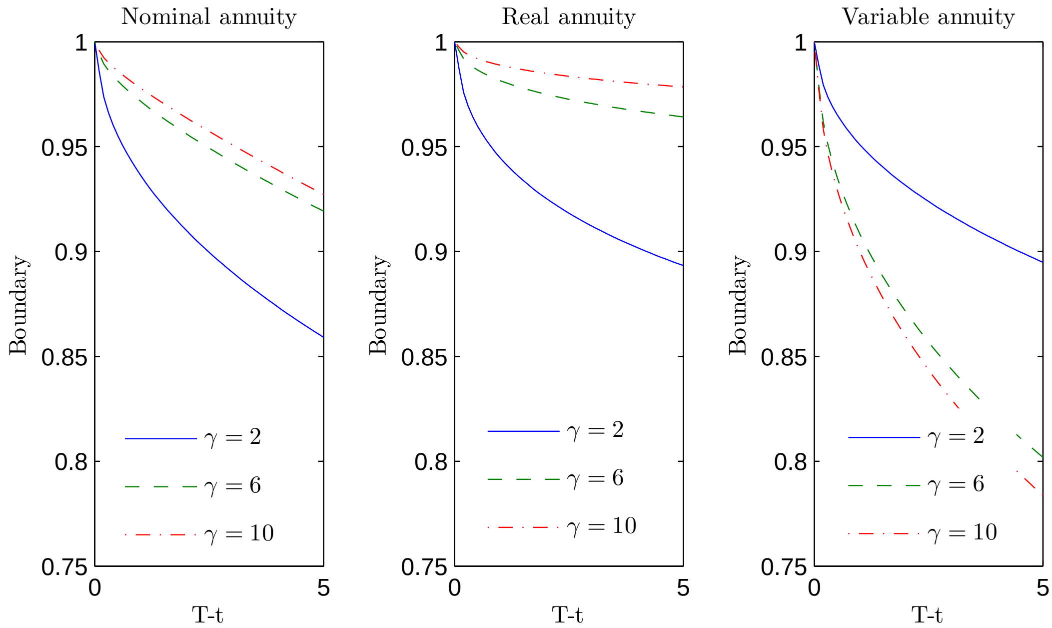

Figure 1: Optimal exercise boundaries. The figure plots the optimal exercise boundaries with different risk aversions for nominal, real , and variable annuities , respectively.

zero. This corresponds to the situation in which, the consumption-to-income ratio hits the boundary from above and liquidity constraints bind. The investor then has to adjust the multiplier so that the consumption-to-income ratio coincides with the value on the boundary. In the subsequent periods, she consumes according to the new multiplier until the boundary is hit again. Put it in another word, the investor exercises the put option on unconstrained wealth whenever the boundary is touched and purchases a new one. The process repeats until the terminal period, in which the investor consumes all remaining wealth.

Figure 1 plots the optimal exercise boundaries for different risk aversion levels for nominal annuities, real annuities, and variable annuities, respectively. In the last period, the boundary equals one, as without bequest motives all of the income is depleted. The boundary is never above one, because liquid wealth is null on the boundary and at that time consumption can only be financed by current income as future income is not marketable. The figure also shows that the boundary is decreasing in the investment horizon. As suggested in Detemple and Serrat (2003), this follows from the fact that immediate exercise of a put option with shorter maturity is always optimal if it is optimal to exercise a longer dated put.

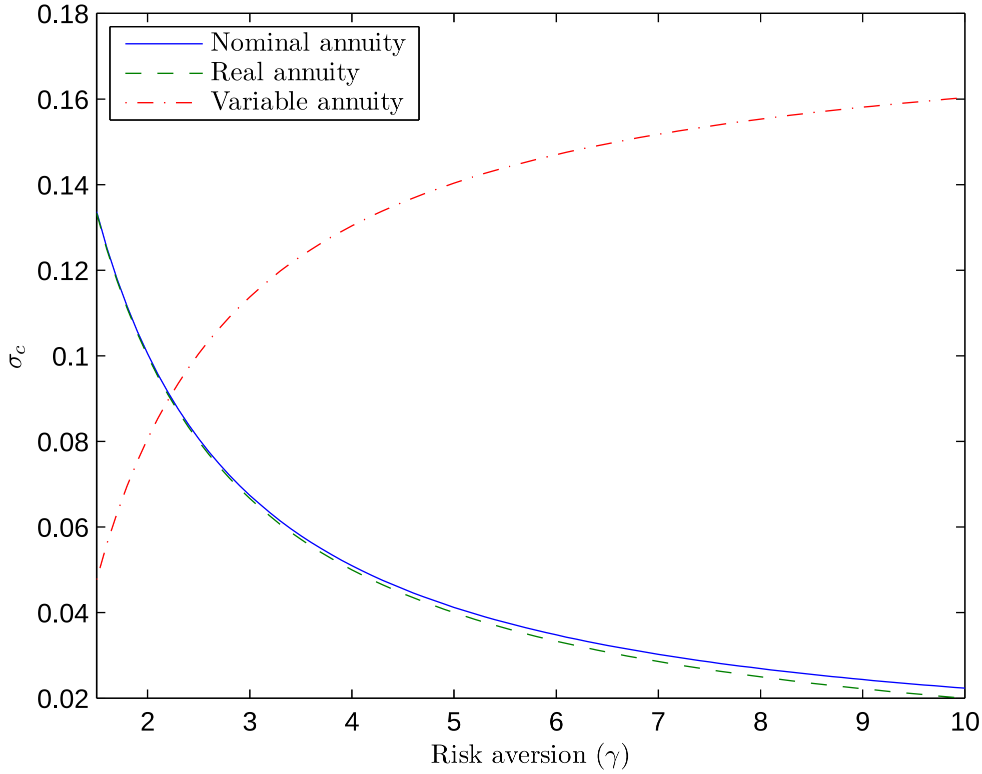

Figure 2: Volatility of consumption-to-income ratio (σ_c). The figure plots the volatility of consumption-to-income ratio (σ_c) along different risk aversions for nominal (solid line), real (dashed line), and variable annuities (dash-dotted line), respectively.

The impact of risk aversion on the boundary is more interesting. Figure 1 illustrates that the boundary is increasing in risk aversion for both nominal and real annuities, and deceasing for variable annuities. This is because for different annuities, risk aversion affects the volatility of consumption-to-income ratio (σ_c) in different ways. Figure 2 plots σ_c as a function of risk aversion (γ) for three annuities. σ_c is decreasing in γ for nominal annuities and real annuities while increasing for variable annuities. As the put option is also written on the consumption-to-income ratio, low (high) volatility of the underlying asset decreases (increases) the downside potential of the put, and decreases (increase) its price.

Table 1: Benchmark Parameter Values

	Financial Market Parameters	
r	Risk-free rate	0.02
σ	Volatility of stock return	0.18
π	Expected inflation	0.03
ξ_s	Inflation innovation parameter	0
ξ_u	Inflation innovation parameter	0.01
ϕ_s	Pricing kernel innovation parameter	-0.2
ϕ_u	Pricing kernel innovation parameter	0
λ_s	Market price of equity risk	0.2
λ_u	Market price of unexpected inflation risk	0.01
ρ_{su}	Correlation between stock and unexpected inflation	0
	Individual Characteristics	
$\hat{W}_0$	Initial Wealth	$\$350,000$
δ	Time preference rate	0.04
γ	Relative risk aversion	[2,10]
	Mortality Parameters	
T	Maximum years to live	35
λ_0	Initial mortality intensity	0.0084
λ_1	Increasing rate of mortality intensity	0.01

4 Optimal annuitization strategies

In this section, we compare the effects of liquidity constraints on different annuity products. We price the annuity products in the financial market described in the previous section. The pricing framework for nominal, real, and variable annuities follows Koijen, Nijman, and Werker (2011). Benchmark parameter values are shown in Table 1. Figure 3 plots the average real (monthly) annuity income provided different annuities. Real annuity has a flat payoff stream. Nominal annuity provides higher payoffs in early periods up to age 75 than real annuity, but its payoffs are decreasing because of the erosion of inflation. The average payoff stream of variable annuity is increasing and higher than that of real annuities for all ages.

At age 65, the retiree decides the fraction of her initial wealth ($\$350,000$) to purchase a certain type of annuity product. Given the annuitization level (π_A), her future income stream and the remaining liquid wealth are determined. Hence, we can apply Theorem 1 in the previous section to solve for the optimal consumption strategy in the presence of non-marketable annuity income. With the optimal consumption strategy, the indirect

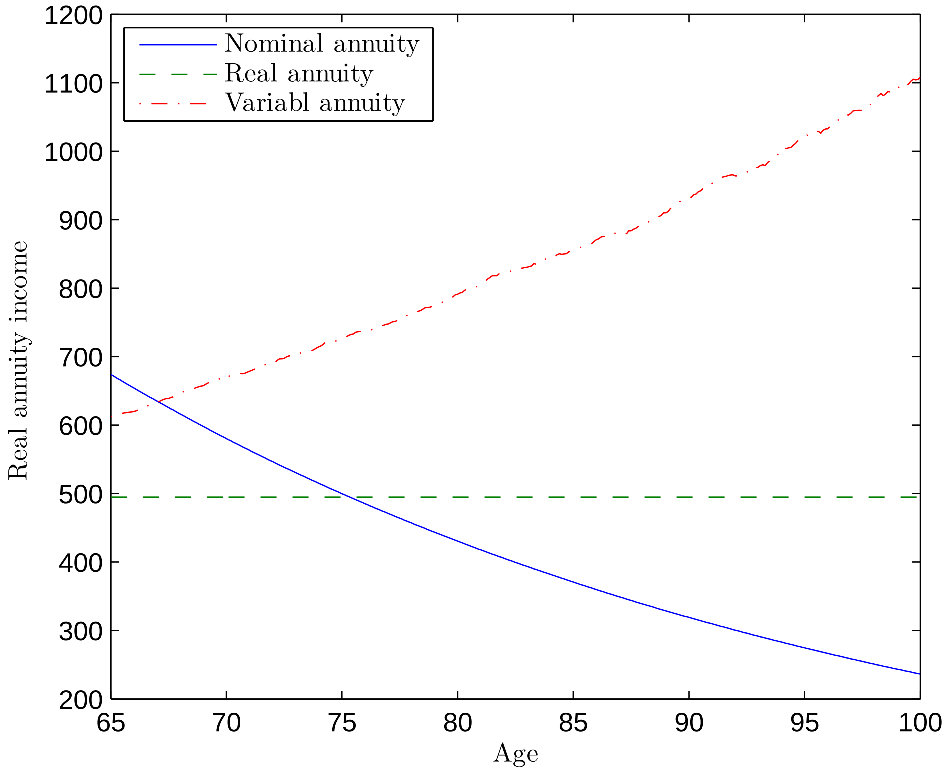

Figure 3: Average real (monthly) annuity payoffs provided by different annuities. The figure plots the average real (monthly) annuity payoffs provides by nominal (solid line), real (dashed line), and variable annuities (dash-dotted line) for \$ 100,000 converted at age 65.

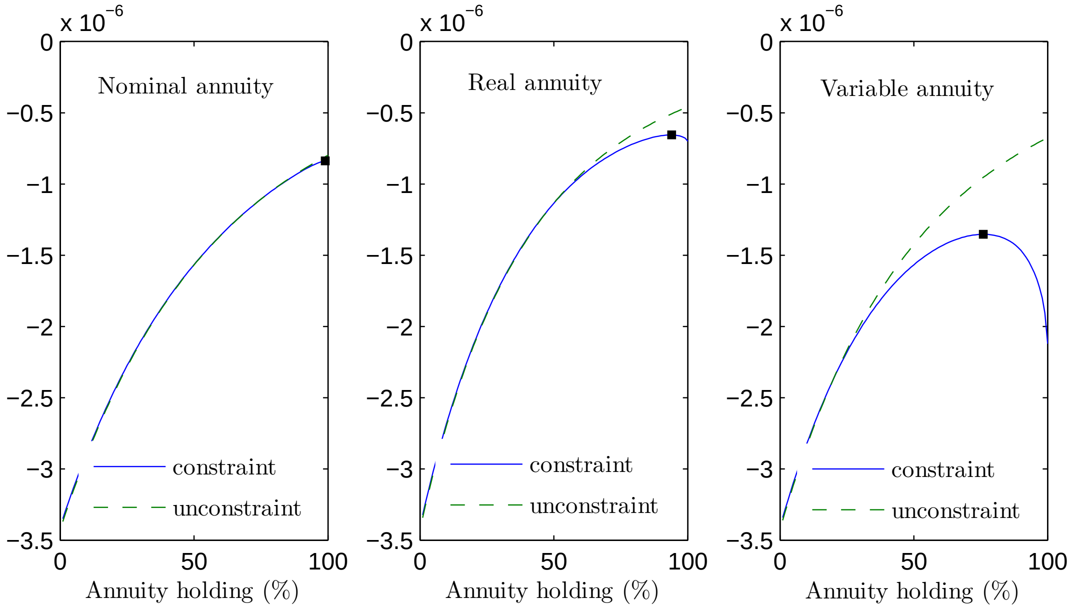

Figure 4: Optimal annuitization levels ($\gamma = 6$). The figure plots the indirect utility at retirement for different annuitization levels with liquidity constraints (solid line) and without liquidity constraints (dashed line), respectively. The solid square represents the optimal annuitization associated with the corresponding annuity product. Risk aversion is 6.

utility at age 65 can be obtained. We then compare the indirect utility associated with different annuitization levels and search for the optimal annuity purchase strategy at retirement over a grid with step size of 1%.

As individual preference plays an important role in determining the effects of liquidity constraints, we illustrate the optimal annuity decisions in the context of high risk aversion as well as low risk aversion. Figure 4 plots the indirect utility associated with different annuitization levels with and without liquidity constraints, for a retiree with relatively high risk aversion ($\gamma = 6$). The unconstrained utility is increasing in annuitization level, while the constrained utility exhibits a hump shaped pattern because of the trade-off between higher mortality credit and stronger liquidity constraints for increasing annuitization levels. Optimal annuitization level under liquidity constraints is still close to 100% for nominal annuities (99%), as the decreasing pattern of payoff stream makes liquidity constraints less binding. In contract, with an increasing payoff stream, optimal annuitization level for variable annuities is only 76%. The impact of liquidity constraints on real annuities is modest, with the optimal annuitization level being 94%.

Figure 5 presents the results for a retiree with lower risk aversion ($\gamma = 2$). Although

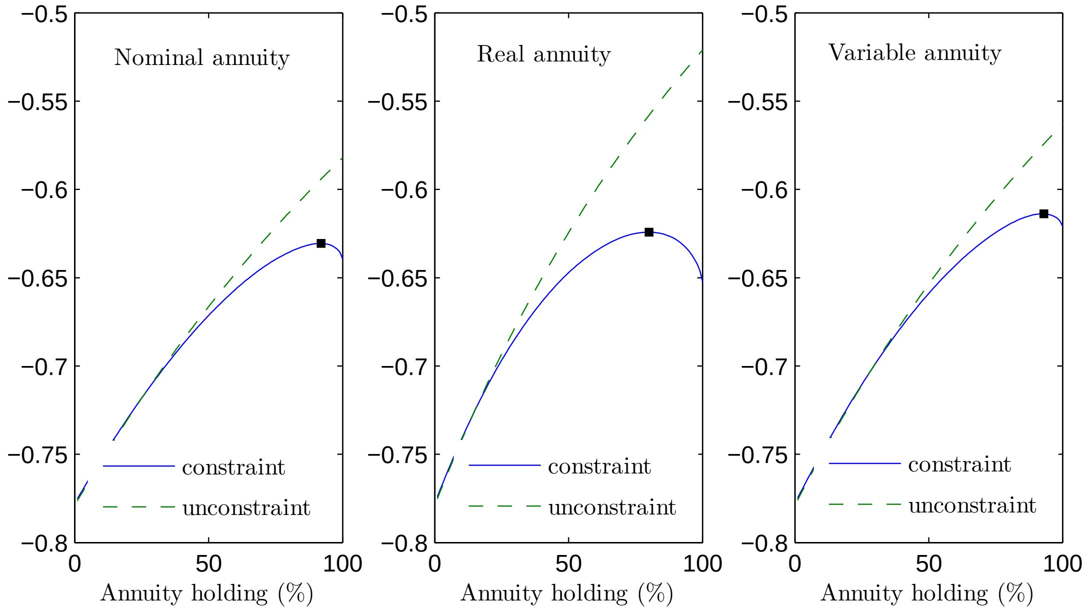

Figure 5: Optimal annuitization levels ($\gamma = 2$). The figure plots the indirect utility at retirement for different annuitization levels with liquidity constraints (solid line) and without liquidity constraints (dashed line), respectively. The solid square represents the optimal annuitization associated with the corresponding annuity product. Risk aversion is 2.

the constrained utility still exhibits a hump-shape pattern for each annuity product, the optimal annualization level is different. Compared to the case with high risk aversion, the demand for real annuities decrease by almost 15% (from 94% to 80%) while the demand for variable annuities increases by more than 20% (from 76% to 93%). The demand for nominal annuities decreases only slightly and the level is still high (92%), reflecting the fact that the impact of liquidity constraints is still small.

To better illustrate the welfare losses caused by liquidity constraints, for each annuity product, we define the certainty equivalent loss (ce) for a given annuitization level as the percentage of wealth lost due to liquidity constraints. In particular, we have

$$J^u_{i,0}(W_0(1-ce);\pi_A) = J^c_{i,0}(W_0;\pi_A) \tag{41}$$

where $J^u_{i,0}(\cdot;\pi_A)$ is the indirect utility for annuity product i $(\in Nominal, Real, Variable)$ at initial time without liquidity constraints given annuitization level π_A, and $J^c_{i,0}(\cdot;\pi_A)$ is the corresponding indirect utility with liquidity constraints.

Figure 6 plots the certainty equivalent loss under different annuitization levels. Given

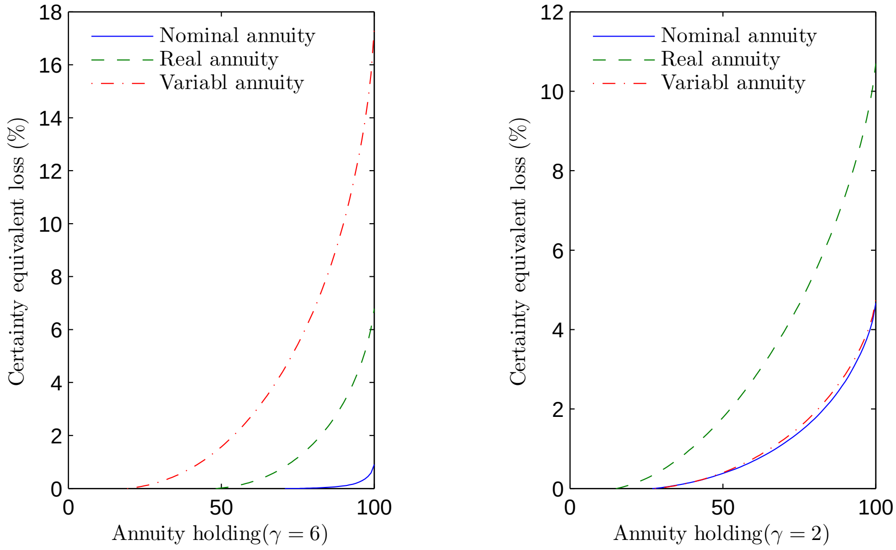

Figure 6: Certainty equivalent loss under different annuitization levels. The figure plots the certainty equivalent loss under different annuitization levels for nominal (solid line), real (dashed line), and variable annuities (dash-dotted line), respectively. The left panel presents the case when risk aversion is 6, the right panel shows the case when risk aversion is 2.

a certain annuity product, the certainty equivalent loss is increasing in annuitization levels. At low annuitization levels, there is almost no utility loss. This is because with higher annuitization levels, initial liquidity wealth is lower while future income is higher. Consequently, were annuity income reversible, the retiree would have borrowed more against future income to smooth consumption. Compare the three annuity products, nominal annuity is least affected by the liquidity constraints, with less than 5% wealth loss. For higher risk aversion, the utility loss is highest for variable annuities. If a retire uses all her initial wealth to buy variable annuities, nearly 18% of the wealth is lost due to liquidity constraints. In contrast, for lower risk aversion, utility loss is highest for real annuities. If a retire uses all her initial wealth to buy real annuities, nearly 12% of the wealth is lost due to liquidity constraints. The relationship between risk aversion and different annuity demands can be interpreted from two perspective. First, recall the discussion about the impacts of risk aversion on boundaries in section 3.2.3. The exercise boundary is decreasing (increasing) in risk aversion for variable annuities (nominal and real annuities), which makes liquidity constraints easier (more difficult) to bind, and consequently decreases (increase) the demand for annuities. Second, note that the impacts of liquidity constraints depend on the retiree's behaviour in the absence of

Table 2: Optimal annuitization levels.
This table shows the optimal annuitization levels for different combinations of risk aversion and purchase time. Other parameter values can be seen in Table 1.

	Age=55	Age=60	Age=65	Age=70	Age =75	Age=80
Nominal annuity						
$\gamma = 2$	85%	89%	91%	92%	94%	95%
$\gamma = 6$	98%	99%	99%	99%	99%	99%
$\gamma = 8$	99%	99%	99%	99%	99%	99%
$\gamma = 10$	99%	99%	99%	99%	99%	100%
Real annuity						
$\gamma = 2$	69%	74%	79%	83%	88%	90%
$\gamma = 6$	89%	91%	93%	94%	96%	97%
$\gamma = 8$	92%	93%	95%	96%	97%	98%
$\gamma = 10$	94%	95%	96%	97%	98%	98%
Variable annuity						
$\gamma = 2$	87%	90%	92%	94%	95%	97%
$\gamma = 6$	65%	71%	76%	78%	84%	87%
$\gamma = 8$	62%	68%	73%	77%	82%	85%
$\gamma = 10$	60%	66%	71%	76%	80%	83%

constraints. If the retiree already prefers assets with both higher expected payoff and volatility (when aversion is low), variable annuity is more attractive as its payment patterns mimics payoff from stocks. On the other hand, nominal and real annuities are comparable to less risky bond products, which are preferred by people with higher risk aversion.

If annuity income is marketable, full annuitization is always optimal. However, with liquidity constraints, the optimal annuity choices now depend on the preference of the retiree as well as the level of mortality credit. Table 2 presents the optimal annuitization level for different combinations of risk aversion and age. Optimal annuitization level is increasing in age. This relationship is driven by two factors. First, mortality credit is higher at older ages. Second, as time horizon decreases, liquidity constraints are also less binding. At age 80, optimal annuitization levels are all above 80%. The effects of risk aversion is more striking. For nominal annuities and real annuities, the desired amount is higher for averse people with higher risk aversion. In particular, real annuity demand can be as less as 69%. In contrast, variable annuity demand is decreasing in risk aversion.

5 Conclusion

In this paper we solve the optimal consumption strategy for a retiree with illiquid annuity income, inflation risk, and asset menu outside annuities in closed form. Based on that, we investigates the liquidity constraints on different annuity products. Full annuitization is in general not optimal when future annuity income is not marketable. The optimal annuity holding depends both on the feature of the income process and on the preference of the retiree. In general, nominal annuities suffer the least from liquidity constraint, as they have higher payoff in early periods. The demand for real annuities increases in risk aversion, as real annuities mimic risk-free assets. On the contrary, the demand for variable annuities decreases in risk aversion.

In recent years an increasing number of financial instruments are available for use in retirement saving, both within and outside the family of annuities. Meanwhile, the transformation from defined-benefit to defined-contribution pension plans in many countries requires careful and active management of retirement wealth. Thus, it's important for the retiree to understand various aspects of a product and make decisions tailored to his own preference. Our paper focus on one important aspect of financial instruments: the level liquidity constraints. We show that liquidity constraints affect the demand for annuities products, which also implies that the demand for liquid financial assets outside the annuity menu changes. The impact of liquidity constraints depend on both the features of different annuity products as well as the retiree's preference. We believe that future works that further investigate the roles of different financial instruments in retirement are promising.

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Appendices

6 Proof of Theorem 1

The unconstrained wealth process w_t^u is

$$\begin{aligned} w_t^u &= \mathrm{E}\left[\int_t^T \frac{m_s}{m_t}\left(\psi \mathrm{e}^{\int_0^s \hat{\delta}_u \mathrm{d}u} m_s\right)^{-\frac{1}{\gamma}} \mathrm{d}s\right] - \mathrm{E}_t\left[\int_t^T \frac{m_s}{m_t} y_s \mathrm{d}s\right] \\ &= c_t^u \int_t^T \mathrm{e}^{-\int_t^s \frac{\hat{\delta}_u}{\gamma}\mathrm{d}u} \mathrm{E}_t\left[\left(\frac{m_s}{m_t}\right)^{1-\frac{1}{\gamma}}\right] \mathrm{d}s - y_t \int_t^T \mathrm{E}_t\left[\frac{m_s}{m_t} y_s\right] \mathrm{d}s \\ &= c_t^u G_t - y_t F_t, \end{aligned} \tag{42}$$

where c_t^u and y_t are unconstrained real consumption process and real annuity income, respectively.

It can easily be verified that c_t^u and y_t evolve as,

$$\mathrm{d}c_t^u = \frac{c_t^u}{\gamma}\left\{\left[-\hat{\delta}_t + r + \frac{1}{2}\left(\frac{1}{\gamma} - 1\right)\phi' \rho \phi\right] \mathrm{d}t - \phi' \mathrm{d}z_t^Q\right\} \tag{43}$$

The early exercise premium representation gives

$$\Omega\left(c_t^u, y_t; B(\cdot)\right) = \mathrm{E}_t^Q\left[\int_t^T \mathrm{e}^{-r(v-t)}\left(y_v - c_v^u\right) 1_{\left\{\frac{c_v^u}{y_v} \leq B_v^*\right\}} \mathrm{d}v\right] \tag{44}$$

where $B_v^* = \frac{B_v^{-\frac{1}{\gamma}}}{y_v}$. Passing to the new measure

$$\mathrm{d}Q^y = \mathrm{e}^{\left(-\frac{1}{2}\sigma_y' \rho \sigma_y T + \sigma_y' z_T^Q\right)} \mathrm{d}Q \tag{45}$$

yields,

$$\Omega\left(y_t c_t^*, y_t; B(\cdot)\right) = y_t \mathrm{E}_t^y\left[\int_t^T \mathrm{e}^{-(r-\mu_y)(v-t)}\left(1 - c_v^*\right) 1_{\{c_v^* \leq B_v^*\}} \mathrm{d}v\right] \tag{46}$$

where $c_v^* = c_v^u / y_v$ satisfies,

$$\frac{\mathrm{d}c_t^*}{c_t^*} = \mu^* \mathrm{d}t - \left(\sigma_y' + \frac{\phi'}{\gamma}\right) \mathrm{d}z_t^y, \tag{47}$$

where μ^* is given by,

$$\mu^* = \frac{1}{\gamma}\left[-\hat{\delta}_t + r + \frac{1}{2}\left(\frac{1}{\gamma} - 1\right)\phi' \rho \phi\right] - \mu_y. \tag{48}$$

Since the linear combination of the component of a normal random vector is normally distributed and normal distribution is symmetric , we have

$$\begin{aligned} -\left(\sigma_y' + \frac{\phi'}{\gamma}\right) \mathrm{d}z_t^y &= \sqrt{\left(\sigma_y + \frac{\phi}{\gamma}\right)' (\rho. \times \rho) \left(\sigma_y + \frac{\phi}{\gamma}\right)} \mathrm{d}\tilde{z}_t^y \\ &\equiv \sigma_c \mathrm{d}\tilde{z}_t^y \end{aligned} \tag{49}$$

where $\tilde{z}_t^y$ is a one-dimensional standard Brownian motion and $\rho. \times \rho$ is the entry-by-entry multiplication.

Thus, c_t^* is a geometric Brownian motion with a time-varying drift term.

$$c_v^* = c_t^* \exp\left\{\int_t^v \left(\mu_s^* - \frac{1}{2}\sigma_c^2\right) \mathrm{d}s - \int_t^v \sigma_x \mathrm{d}z_s^y\right\} \tag{50}$$

The event $\{c_v^* \leq B_v\}$ is equivalent to

$$\begin{aligned} \frac{z_v^y - z_t^y}{\sqrt{v-t}} &\leq -\frac{\log(c_t^*/B_v) + \int_t^v \left(\mu_s^* - \frac{1}{2}\sigma_c^2\right) \mathrm{d}s}{\sigma_c \sqrt{v-t}} \\ &\equiv -d(c_t^*, B_v, t, v) + \sigma_c \sqrt{v-t} \end{aligned} \tag{51}$$

where

$$d(c_t^*, B_v, t, v) = \frac{\log(c_t^*/B_v) + \int_t^v \left(\mu_s^* + \frac{1}{2}\sigma_c^2\right) \mathrm{d}s}{\sigma_c \sqrt{v-t}} \tag{52}$$

We have

$$\Omega\left(y_t c_t^*, y_t; B(\cdot)\right) = y_t \left[\int_t^T \mathrm{e}^{(\mu_y - r)(v-t)} \Phi\left(c_t^*, B_v, t, v\right) \mathrm{d}v\right] \tag{53}$$

where

$$\begin{aligned}\Phi\left(c_t^*, B_v, t, v\right) =& N\left(-d(c_t^*, B_v, t, v) + \sigma_c\sqrt{v-t}\right)\\ &- z_t^y \mathrm{e}^{\int_t^v \mu_s^* \mathrm{d}s} N\left(-d(c_t^*, B_v, t, v)\right)\end{aligned} \tag{54}$$

Define a new notation

$$\begin{aligned}\Omega\left(c_t^*; B(\cdot)\right) &= \Omega\left(c_t^*, 1; B(\cdot)\right)\\ &= \left[\int_t^T \mathrm{e}^{(\mu_y - r)(v-t)} \Phi\left(c_t^*, B_v, t, v\right) \mathrm{d}v\right]\end{aligned} \tag{55}$$

The optimal constrained wealth is

$$w_t = y_t V(c_t^*, t) + y_t \Omega\left(c_t^*; B(\cdot)\right) \tag{56}$$

where

$$V(c_t^*, t) = c_t^* G_t - F_t \tag{57}$$

Since $w_t^u(c_t^*) = y_t\left(c_t^* G_t - F_t\right)$,the boundary B solves the recursive integral equation

$$\Omega\left(B_t; B(\cdot)\right) = F_t - B_t G_t \tag{58}$$

with terminal condition $B_T = 1$.

By applying Theorem 3.3 and Propositions 3,5 and 3.6 in El Karoui and Jeanblanc-Picqué (1998) and Theorem 1 in Detemple and Serrat (2003), we have:

Optimal initial consumption c_0^* is given by

$$y_0 V\left(\frac{c_0^*}{y_0}, t\right) + y_0 \Omega\left(\frac{c_0^*}{y_0}; B(\cdot)\right) = w_0 \tag{59}$$

Optimal initial multiplier ψ^* is given by

$$c_0^* = (\psi^* m_0)^{-\frac{1}{\gamma}} \tag{60}$$

Optimal real consumption c_t^* is

$$c_t^* = \left((\psi^* - \varsigma_t^*)\mathrm{e}^{\int_0^t \hat{\delta}_u \mathrm{d}u} m_t\right)^{-\frac{1}{\gamma}} \tag{61}$$

where

$$\varsigma_t^* = \left(\psi^* - \inf_{v\in[0,t]} \frac{(B_v y_v)^{-\gamma}}{\mathrm{e}^{\int_0^v \hat{\delta}_u \mathrm{d}u} m_v}\right)^+ \tag{62}$$